Existence and Uniqueness of Inductive Limit Cartan Subalgebras in Inductive Limit C^*-algebras

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Cartan Subalgebras - Why We Care

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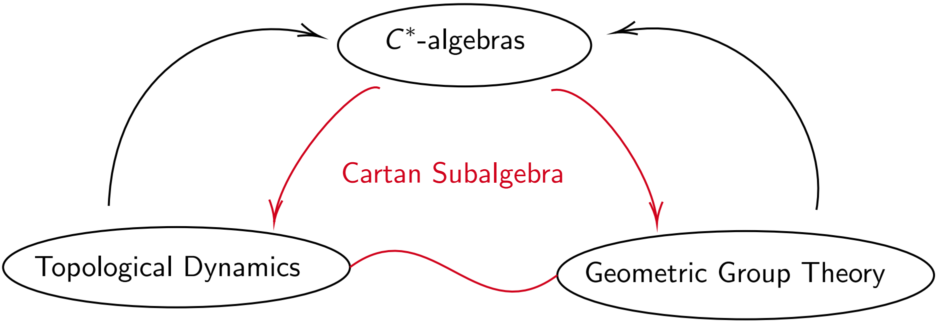

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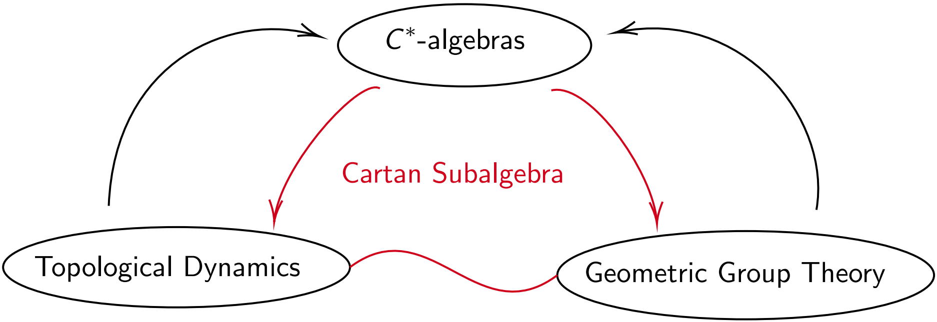

UCT Problem $\rightsquigarrow$ Existence of Cartan subalgebras

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In general, $C_0(\mathcal{G}^0) \subset C_r^*(\mathcal{G}, \Sigma)$

Recent Interest - Existence and Uniqueness

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Li & Renault: *Cartan Subalgebras in C^*-algebras. Existence and Uniqueness.* 2019

Li: *Every Classifiable Simple C^*-algebra Has a Cartan Subalgebra*. 2020

White & Willett: *Cartan Subalgebras in Uniform Roe Algebras*. 2020

Barlak & Raum: *Cartan Subalgebras in Dimension Drop Algebras*. 2021

Cartans in Inductive Limits

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Inductive Limit C^*-algebra:

$$\begin{tikzcd} A_1 \arrow[r, "\phi_1"] & A_2 \arrow[r, "\phi_2"] & A_3 \arrow[r, "\phi_3"] & \cdots A_n \arrow[rr, bend left, "\mu_n"] & \cdots & A \end{tikzcd}$$

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$$A_1 \xrightarrow{\phi_1} A_2 \xrightarrow{\phi_2} A_3 \xrightarrow{\phi_3} \cdots A_n \quad \cdots \quad A$$

$$A_n \xrightarrow{\mu_n} A$$

ϕ_n: *connecting map*, A_n: *building block*

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When is C a Cartan subalgebra of A? If it is it would be an *inductive limit Cartan subalgebra.*

Cartans in AF-Algebras

$$M_2(\mathbb{C}) \to M_4(\mathbb{C}) \to M_8(\mathbb{C}) \to \cdots CAR$$

$$A \to \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$$

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Theorem (Stratila-Voiculescu, 1975)

D is a Cartan subalgebra of CAR. Every unital AF-algebra has an inductive limit Cartan subalgebra.

Cartans in AF-Algebras

Note that each connecting map $\phi_n : M_{2^n}(\mathbb{C}) \to M_{2^{n+1}}(\mathbb{C})$ is:

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$P_{n+1} \circ \phi_n = \phi_n \circ P_n$ (conditional expectation compatible with connecting maps)

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Assume $A_1 \xrightarrow{\phi_1} A_2 \xrightarrow{\phi_2} A_3 \xrightarrow{\phi_3} \cdots A_n \xrightarrow{\mu_n} \cdots A$ with injective connecting maps.

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Let $C_n \subset A_n$ a Cartan subalgebra.

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Assume $A_1 \xrightarrow{\phi_1} A_2 \xrightarrow{\phi_2} A_3 \xrightarrow{\phi_3} \cdots A_n \xrightarrow{\mu_n} \cdots A$ with injective connecting maps.

Let $C_n \subset A_n$ a Cartan subalgebra. Then

Theorem (Li, 2020)

If the connecting maps satisfy:

$$\phi_n(C_n) \subset C_{n+1}, \;\; \phi_n(N_{A_n}(C_n)) \subset N_{A_{n+1}}(C_{n+1}), \;\; P_{n+1} \circ \phi_n = \phi_n \circ P_n,$$

then $C_1 \xrightarrow{\phi_1} C_2 \xrightarrow{\phi_2} C_3 \xrightarrow{\phi_3} \cdots C_n \xrightarrow{\mu_n} \cdots C$
C is a Cartan subalgebra of A.

Uniqueness of Inductive Limit Cartan Subalgebras

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If C is an inductive limit Cartan subalgebra of an inductive limit C^*-algebra A, then it is *unique* if

whenever D is an inductive limit Cartan subalgebra of an inductive limit C^*-algebra B and $A \cong B$, then we have an isomorphism

$$(A, C) \cong (B, D)$$

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Question: Are inductive limit Cartan subalgebras of AF-algebras unique?

Uniqueness of AF-Cartan Subalgebras

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Theorem (R.)

Let A and B be unital AF-algebras, with inductive limit Cartan subalgebras C and D, respectively.

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Let A and B be unital AF-algebras, with inductive limit Cartan subalgebras C and D, respectively. Then:

$$(K_0(A), K_0(A)^+, [1_A]_0) \overset{\alpha}{\cong} (K_0(B), K_0(B)^+, [1_B]_0)$$

$$\Longleftrightarrow$$

$$\exists \phi \;\; A \overset{\phi}{\cong} B, \;\; \phi(C) = D, \;\; \textit{and} \;\; K_0(\phi) = \alpha$$

AI, A$\mathbb{T}$, and AX-algebras

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Question: Do AX-algebras have inductive limit Cartan subalgebras?

Existence of Inductive Limit Cartan Subalgebras in A$\mathbb{X}$-Algebras

Existence of Inductive Limit Cartan Subalgebras in AX-Algebras

A *standard connecting map* $\phi : \bigoplus_{j=1}^{N} M_{n_j}(C(X)) \to \bigoplus_{i=1}^{M} M_{m_i}(C(X))$ looks like (on summands)

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$$A \to \begin{pmatrix} A \circ g_1 & & & \\ & A \circ g_2 & & \\ & & \ddots & \\ & & & A \circ g_k \end{pmatrix}$$

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$g_s : X \to X$ called *eigenvalue function*

Existence of Inductive Limit Cartan Subalgebras in AX-Algebras

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3. Every unital AX-algebra with injective connecting maps, where X is a finite connected graph imbedded in $\mathbb{C}$, can be realized using injective connecting maps that are unitary conjugates (over $[0,1]$) of a standard connecting map.

Existence of Inductive Limit Cartan Subalgebras in AX-Algebras

Theorem (R.)

Let X be a finite connected graph imbedded in $\mathbb{C}$, and

$$\phi : \bigoplus_{j=1}^{N} M_{n_j}(C(X)) \to \bigoplus_{i=1}^{M} M_{m_i}(C(X))$$ *a connecting map between AX building blocks.*

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$$\phi(C) \subset D, \ \ \phi(N(C)) \subset N(D)$$

and ϕ is compatible with the conditional expectations.

Existence of Inductive Limit Cartan Subalgebras in AX-Algebras

Theorem (R.)

Every unital AX-algebra (for X a finite connected graph imbedded in $\mathbb{C}$) has an inductive limit Cartan subalgebra. Hence AI and A$\mathbb{T}$-algebras have inductive limit Cartan subalgebras.

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What about Uniqueness?

Theorem (R.)

Every unital simple AI-algebra with injective connecting maps admits two non-isomorphic inductive limit Cartan subalgebras. Uniqueness fails!

Thank You for Listening