Groupoid Correspondences and the ABC Spectral Sequence

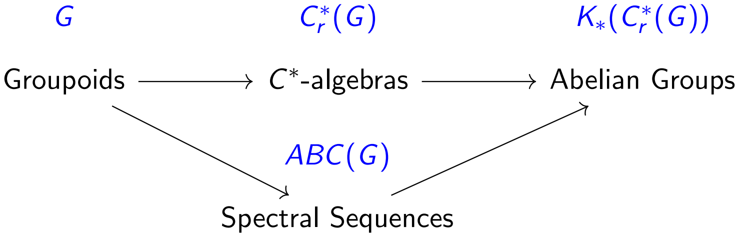
Alistair Miller

Queen Mary University of London

August 9, 2021

Overview - what am I talking about?

I'm interested in understanding the following constructions.



The ABC spectral sequence is a homological tool which helps us to understand the K-theory groups $K_i(C_r^*(G))$.

Question

How can we turn these constructions into functors - which kind of groupoid morphisms give rise to morphisms of C^*-algebras and spectral sequences?

Answer: Groupoid correspondences.

Motivation

Why study the constructions $G \mapsto C_r^*(G) \mapsto K_*(C_r^*(G))$?

- Groupoid C^*-algebras are everywhere.
- K-theory is important for C^*-algebraists.
- We want to understand these better on the groupoid level.

Lots of C^*-algebras are generated by partial isometries, e.g. (higher rank) graph algebras and semigroup C^*-algebras, and relate to inverse semigroups S. $C_r^*(S)$ is a groupoid C^*-algebra to which we can apply our tools to understand its K-theory:

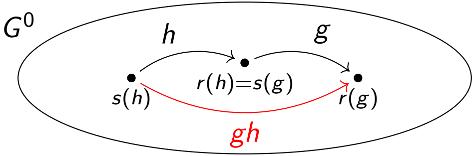
Theorem (M.)

Under conditions on the inverse semigroup S with stabiliser groups $\Gamma_i \subseteq S$,

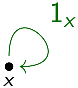
$$K_*(C_r^*(S)) \cong \bigoplus_{i \in I} K_*(C_r^*(\Gamma_i)).$$

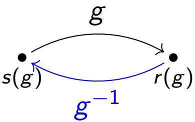
Topological Groupoids

What is a groupoid? We have a topological space G^0 of units and a space G of arrows on the unit space. Each arrow $g \in G$ has a source and a range in G^0.



We can compose arrows when their range and source match up. We also have:





an identity 1_x for each $x \in G^0$, and an inverse g^{-1} for each $g \in G$.

$$1_x g = g = g 1_x \qquad g^{-1} g = 1_{s(g)}, \; g g^{-1} = 1_{r(g)}$$

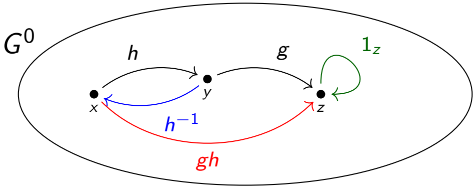
Topological Groupoids

Structure

Unit space G^0

Arrow space G

Range and source



Axioms

Composition

Identity

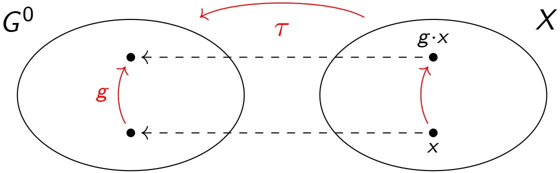
Inversion

Examples

- If there is only one unit $G^0 = \{*\}$, composition is always defined and G is a (topological) group.
- If the only arrows are identity arrows, $G = G^0$ is a topological space.
- If a group acts on a space $\Gamma \curvearrowright X$, we can form the transformation groupoid $\Gamma \ltimes X$ - arrows are triples (x, γ, y) with $\gamma \cdot y = x$.
- An equivalence relation $\sim$ on a space X gives us a groupoid $R_\sim$. We draw $x \to y$ if $y \sim x$.

Actions of Topological Groupoids

How does a groupoid G act on a space X?
Define $g \cdot x$ only if $s(g) = \tau(x)$: need an anchor map $\tau : X \to G^0$. We write $G \curvearrowright X$.



Examples

- Every groupoid G acts on itself $G \curvearrowright G$ by left multiplication ($g \cdot h = gh$) with anchor map $r : G \to G^0$.
- Given an equivalence relation $(X, \sim)$, $R_\sim \curvearrowright X$ by $(y \sim x) \cdot x = y$.
- An "action" of a space $Y \curvearrowright X$ is just an anchor map $X \to Y$.
- An equivariant map $Y \to X$ of Γ-spaces gives us $\Gamma \ltimes X \curvearrowright Y$.

Étale groupoids

Disclaimer

Our topological groupoids will always be étale, locally compact and Hausdorff. We probably also want to say second-countable.

Definition (Étale groupoids)

A groupoid G is étale if the range map $r : G \to G^0$ is a local homeomorphism and the unit space $G^0 \subseteq G$ is open.

Think of this as an analogue of discrete groups rather than locally compact groups.

$$\Gamma \text{ discrete group, } X \text{ any space } \implies \Gamma \ltimes X \text{ étale}$$

Groupoid Correspondences

Definition (Groupoid Correspondence (Holkar))

A correspondence of étale groupoids from G to H is a space Ω with a left G action and a right H action

$$G \curvearrowright \Omega \curvearrowleft H$$

such that:

- The actions commute: $(g \cdot \omega) \cdot h = g \cdot (\omega \cdot h)$ whenever $g \cdot \omega$ and $\omega \cdot h$ are defined.
- The right action $\Omega \curvearrowleft H$ is free and proper.
- The right anchor map $\sigma : \Omega \to H^0$ is a local homeomorphism.

We say $G \curvearrowright \Omega \curvearrowleft H$ is proper if the induced map $\overline{\rho} : \Omega / H \to G^0$ is proper.

We want to think of $G \curvearrowright \Omega \curvearrowleft H$ as a morphism $\Omega : G \to H$. The isomorphisms are Morita equivalences - these give Morita equivalences of C^*-algebras (Muhly, Renault, Williams).

Groupoid Correspondences

Examples (Groupoid correspondences $G \curvearrowright \Omega \curvearrowleft H$)

- A group homomorphism $\phi : G \to H$ gives us actions

$$G \curvearrowright H \curvearrowleft H, \qquad g \cdot h = \phi(g)h, \qquad h \cdot h' = hh'.$$

- A continuous map $f : Y \to X$ of topological spaces gives us actions

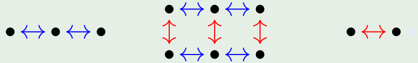
$$X \curvearrowright Y \curvearrowleft Y \quad = \quad X \xleftarrow{f} Y \xrightarrow{\text{id}} Y.$$

- For any groupoid G, we have actions by left and right multiplication

$$G \curvearrowright G \curvearrowleft G.$$

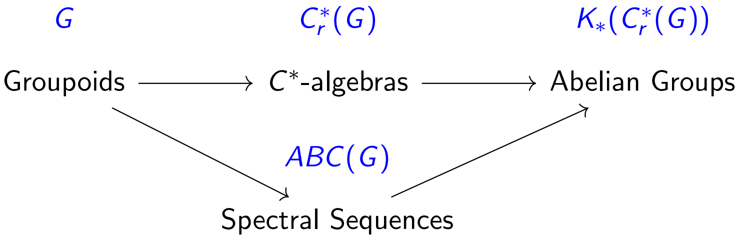
- Given full equivalence relations R_m and R_n on m and n points, we get a correspondence

$$R_m \quad \curvearrowright \quad m \times n \quad \curvearrowleft \quad R_n.$$



Functors with respect to groupoid correspondences

If we take our groupoid morphisms to be proper correspondences, we can make the following constructions functorial.



Theorem (M.)

A proper groupoid correspondence $G \curvearrowright \Omega \curvearrowleft H$ induces a morphism $ABC(G) \to ABC(H)$ of spectral sequences so that $G \mapsto ABC(G)$ is functorial.

What is this “ABC Spectral Sequence”?

Spectral sequences are a homological tool for relating different abelian groups. Proietti and Yamashita applied Meyer’s general/abstract ABC spectral sequence to groupoids.

Theorem (Proietti, Yamashita ’20)

Let G be an étale groupoid with torsion-free stabilisers and totally disconnected unit space that satisfies a strong version of Baum-Connes. Then the groupoid homology $H_*(G)$ “converges to” the K-theory $K_*(C_r^*(G))$ in the ABC spectral sequence for G.

Groupoid	Groupoid Homology	K-theory

$$
\begin{array}{ccc}
G & H_*(G) \xRightarrow{\;ABC(G)\;} & K_*(C_r^*(G)) \\
\Big\downarrow{\scriptstyle G \curvearrowright \Omega \curvearrowleft H} & \Big\downarrow & \Big\downarrow \\
H & H_*(H) \xRightarrow{\;ABC(H)\;} & K_*(C_r^*(H))
\end{array}
$$

Application: K-theory of an inverse semigroup C*-algebra

An inverse semigroup S is an object like a group that relates to partial symmetries. There is an étale "universal" groupoid $G(S)$ associated to it.

Theorem (M.)

Under conditions on the inverse semigroup S with stabiliser groups $\Gamma_i \subseteq S$,

$$K_*(C_r^*(S)) \cong K_*(C_r^*(G(S))) \cong \bigoplus_{i\in I} K_*(C_r^*(\Gamma_i)).$$

Proof: We can construct a groupoid correspondence $\amalg_{i\in I}\Gamma_i \to G(S)$.

$$\begin{array}{ccc} H_*(\amalg_{i\in I}\Gamma_i) & \xRightarrow{ABC(\amalg_{i\in I}\Gamma_i)} & K_*(C_r^*(\amalg_{i\in I}\Gamma_i)) \\ \downarrow\cong & & \downarrow\cong \\ H_*(G(S)) & \xRightarrow{ABC(G(S))} & K_*(C_r^*(G(S))) \end{array}$$

After applying our functor, we can demonstrate an isomorphism in groupoid homology and therefore deduce an isomorphism in K-theory. □

Thank you for listening!

Key References

- A. Sims - Hausdorff étale groupoids and their C^*-algebras.
- R. D. Holkar - Topological construction of C^*-correspondences for groupoid C^*-algebras.
- S. Albandik - A colimit construction for groupoids.
- R. Meyer - Homological algebra in bivariant K-theory and other triangulated categories II.
- V. Proietti and M. Yamashita - Homology and K-theory of torsion-free ample groupoids and Smale spaces.
- P. S. Muhly, J. N. Renault and D. P. Williams - Equivalence and isomorphism for groupoid C^*-algebras.
- P. S. Muhly and D. P. Williams - Renault's Equivalence Theorem for Groupoid Crossed Products.