C*-like Modules.

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Hilbert Modules

Roughly speaking, a Hilbert module is like a Hilbert space but the scalars are in a C*-algebra.

Definition

Let A be a C*-algebra. A (right) **Hilbert A-module** is a complex vector space X which is a right A-module with an A-valued right inner product

$$\begin{array}{rcl} \mathsf{X} \times \mathsf{X} & \to & A \\ (x, y) & \mapsto & \langle x, y \rangle_A \end{array}$$

such that X is complete with the induced norm

$$\|x\| := \|\langle x, x \rangle_A\|^{1/2}.$$

A-valued right inner product.

Definition

Let A be a C*-algebra and X a complex vector space which is also a right A-module. An A-**valued right inner product on** X is a map

$$\begin{array}{ccc} \mathsf{X} \times \mathsf{X} & \to & A \\ (x, y) & \mapsto & \langle x, y \rangle_A \end{array}$$

such that for any $x, y, y_1, y_2 \in \mathsf{X}$, $a \in A$ and $\alpha \in \mathbb{C}$ we have

1. $\langle x, y_1 + \alpha y_2 \rangle_A = \langle x, y_1 \rangle_A + \alpha \langle x, y_2 \rangle_A$.
2. $\langle x, ya \rangle_A = \langle x, y \rangle_A a$.
3. $\langle x, y \rangle_A^* = \langle y, x \rangle_A$.
4. $\langle x, x \rangle_A \geq 0$ in A.
5. $\langle x, x \rangle_A = 0 \Longrightarrow x = 0$.

Hilbert Modules: Examples

Example

Let $\mathcal{H}$ be a Hilbert space with the physicists' convention that the inner product is linear in the second variable. Then, $\mathcal{H}$ is a Hilbert $\mathbb{C}$-module.

Example

Any C*-algebra A is a Hilbert A-module with action given by multiplication and inner product given by $(a, b) \mapsto a^*b$.

Example

Let X and Y be HIlbert A-modules. Then X $\oplus$ Y is a Hilbert A-module with coordinate wise action and inner product given by

$$\langle (x_1, y_1), (x_2, y_2) \rangle = \langle x_1, x_2 \rangle + \langle y_1, y_2 \rangle.$$

Adjointable maps

A main difference between Hilbert modules and Hilbert spaces is that not every bounded linear map between Hilbert A-modules has an adjoint.

Definition

Let X and Y be Hilbert A-modules. A map $t : \mathsf{X} \to \mathsf{Y}$ is said to be **adjointable** if there is a map $t^* : \mathsf{Y} \to \mathsf{X}$ such that for any $x \in \mathsf{X}$, and $y \in \mathsf{Y}$

$$\langle t(x), y \rangle_A = \langle x, t^*(y) \rangle_A$$

The space of adjointable maps from X to Y is denoted by $\mathcal{L}_A(\mathsf{X}, \mathsf{Y})$ and $\mathcal{L}_A(\mathsf{X}) := \mathcal{L}_A(\mathsf{X}, \mathsf{X})$.

- Adjointable maps between Hilbert modules are linear, bounded and module maps;
- $\mathcal{L}_A(\mathsf{X})$ is a C*-algebra when equipped with the operator norm.

Generalized Compact Operators

We will have special interest for a particular case of adjointable maps, those of "rank 1":

Definition

Let X and Y be Hilbert A-modules. For $x \in$ X and $y \in$ Y, we define a map $\theta_{x,y} :$ Y $\to$ X by

$$\theta_{x,y}(z) := x\langle y, z\rangle_A$$

One checks that $\theta_{x,y} \in \mathcal{L}_A(\mathsf{Y}, \mathsf{X})$ with $(\theta_{x,y})^* = \theta_{y,x} \in \mathcal{L}_A(\mathsf{X}, \mathsf{Y})$.

Generalized Compact Operators

The maps $\theta_{x,y}$ give an analogue of the of rank-one operators on Hilbert spaces. So, we define an analogue of the compact operators by letting

$$\mathcal{K}_A(\mathsf{Y},\mathsf{X}) := \overline{\mathrm{span}\{\theta_{x,y} : x \in \mathsf{X}, y \in \mathsf{Y}\}}.$$

In fact, $\mathcal{K}_A(\mathsf{X}) := \mathcal{K}_A(\mathsf{X},\mathsf{X})$ is a closed two sided ideal in $\mathcal{L}_A(\mathsf{X})$, whence $\mathcal{K}_A(\mathsf{X})$ is also a C*-algebra.

Furthermore, X is a **left** Hilbert $\mathcal{K}_A(\mathsf{X})$-module with the obvious action and left inner product given by

$${}_{\mathcal{K}_A(\mathsf{X})}\langle x, y\rangle = \theta_{x,y}.$$

The Linking Algebra

The linking algebra of a Hilbert A-module X is

$$\mathbb{L}_{\mathsf{X}} = \begin{pmatrix} \mathcal{K}_A(\mathsf{X}) & \mathsf{X} \\ \widetilde{\mathsf{X}} & A \end{pmatrix} = \left\{ \begin{pmatrix} k & x \\ \tilde{y} & a \end{pmatrix} : k \in \mathcal{K}_A(\mathsf{X}), x, y \in \mathsf{X}, a \in A \right\}$$

- $\widetilde{\mathsf{X}} = \{\tilde{x} \colon x \in \mathsf{X}\}$ is the conjugate vector space of X.
- $\mathbb{L}_{\mathsf{X}}$ is an algebra with the formal matrix multiplication obtained via the actions and inner products.
- Moreover, $\mathbb{L}_{\mathsf{X}}$ is a C*-subalgebra of $\mathcal{L}_A(\mathsf{X} \oplus A)$.

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Let $\mathcal{H}_0, \mathcal{H}_1$ be Hilbert spaces and fix a concrete C*-algebra $A \subseteq \mathcal{L}(\mathcal{H}_0)$.

Main Example

Let $\mathsf{X} \subseteq \mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$ be a closed subspace such that

1. $xa \in \mathsf{X}$ for all $x \in \mathsf{X}$, $a \in A$.
2. $x^* y \in A$ for all $x, y \in \mathsf{X}$.

Then, X is a right Hilbert A-module with A-valued inner product given by $\langle x, y \rangle_A = x^* y$.

Proposition

Suppose that $\operatorname{span}\{x\xi \colon x \in \mathsf{X}, \xi \in \mathcal{H}_0\}$ *is dense in* $\mathcal{H}_1$. *Then*

- $\mathcal{K}_A(\mathsf{X}) \cong \overline{\operatorname{span}\{xy^* \colon x, y \in \mathsf{X}\}} \subseteq \mathcal{L}(\mathcal{H}_1)$ *via* $\theta_{x,y} \mapsto xy^*$.
- $\mathcal{L}_A(\mathsf{X}) \cong \{b \in \mathcal{L}(\mathcal{H}_1) \colon bx, b^*x \in \mathsf{X}$ *for all* $x \in \mathsf{X}\}$ *via* $t \mapsto b_t$, *where* b_t *is determined by* $b_t(x\xi) = t(x)\xi$.

Main Example

Let $\mathsf{X} \subseteq \mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$ be a closed subspace such that $xa \in \mathsf{X}$ and $x^*y \in A$ for all $x, y \in \mathsf{X}$ and $a \in A$. Then, X is a right Hilbert A-module with A-valued inner product given by $\langle x, y \rangle_A = x^*y$.

Theorem (D. 2021)

Let A be a C-algebra and let X be a right Hilbert A-module. Then, there are Hilbert spaces $\mathcal{H}_0$ and $\mathcal{H}_1$ such that the following hold:*

1. *A is faithfully represented on $\mathcal{H}_0$,*
2. *X is isometrically isomorphic to a closed subspace of $\mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$.*

Furthermore, the image of A in $\mathcal{L}(\mathcal{H}_0)$ and the image of X in $\mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$ have the Hilbert module structure from the main example above.

Sketch of Proof. Let $\pi_A\colon A \to \mathcal{L}(\mathcal{H}_0)$ be a non-degenerate faithful representation of A on $\mathcal{H}_0$. Define $\mathcal{H}_1 = \mathsf{X} \otimes_{\pi_A} \mathcal{H}_0$. For each $x \in \mathsf{X}$, the creation operator $c_x : \mathcal{H}_0 \to \mathcal{H}_1$, given by $c_x\xi = x \otimes \xi$, satisfies $\|c_x\| = \|x\|$. Each $k \in \mathcal{K}_A(\mathsf{X})$ induces a map in $\mathcal{L}(\mathcal{H}_1)$ such that $x \otimes \xi \mapsto k(x) \otimes \xi$. This gives rise to a faithful representation $\pi_{\mathcal{K}} : \mathcal{K}_A(\mathsf{X}) \to \mathcal{L}(\mathcal{H}_1)$. Then, we obtain $\pi\colon \mathbb{L}_{\mathsf{X}} \to \mathcal{L}(\mathcal{H}_1 \oplus \mathcal{H}_0)$, a representation of the linking algebra of X, by letting

$$\pi \begin{pmatrix} k & x \\ \tilde{y} & a \end{pmatrix} \begin{pmatrix} \eta \\ \xi \end{pmatrix} = \begin{pmatrix} \pi_{\mathcal{K}}(k)\eta + c_x\xi \\ c_y^*\eta + \pi_A(a)\xi \end{pmatrix}, \ \forall \ \eta \in \mathcal{H}_1, \xi \in \mathcal{H}_0.$$

Finally,

$$\mathsf{X} \cong \pi(\mathsf{X}) \cong \{c_x : x \in \mathsf{X}\} \subseteq \mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$$

isometrically. The desired conclusion now follows. "□"

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L^p-Operator Algebras

Let (Ω, μ) be a measure space and let $p \in [1, \infty)$.

Definition

We say a Banach algebra A is an L^p-*Operator Algebra* if there is an isometric algebra homomorphism $A \hookrightarrow \mathcal{L}(L^p(\Omega, \mu))$.

C*-like Modules

Definition ($(\Omega_0, \mu_0), (\Omega_1, \mu_1)$ are measure spaces and $p \in (1, \infty)$)

Let $A \subseteq \mathcal{L}(L^p(\Omega_0, \mu_0))$ be an L^p operator algebra. A *C*-like module* over A is a pair (X, Y) such that

1. X $\subseteq \mathcal{L}(L^p(\Omega_0, \mu_0), L^p(\Omega_1, \mu_1))$ is a closed subspace,
2. Y $\subseteq \mathcal{L}(L^p(\Omega_1, \mu_1), L^p(\Omega_0, \mu_0))$ is a closed subspace,
3. $xa \in$ X for all $x \in$ X and $a \in A$,
4. $yx \in A$ for all $x \in$ X and $y \in$ Y.

Since Y plays the role of X*, it seems reasonable to also ask

5. $ay \in$ Y for all $y \in$ Y, $a \in A$.
6. The norm in X is determined by the A-valued pairing:

$$\|x\| = \sup_{\|y\|=1} \|yx\|.$$

Example I

Let $p \in (1, \infty)$. For $d \in \mathbb{Z}_{\geq 1}$ we set $\ell_d^p = \ell^p(\{1, \ldots, d\})$.

Example

If A is any L^p operator algebra, then (A, A) is a C*-like module satisfying ❶ through ❺. Condition ❻ fails in general. However, if A has a contractive approximate identity, ❻ holds.

An instance for which ❻ fails is when

$$A = \left\{ \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix} : z \in \mathbb{C} \right\} \subset M_2^p(\mathbb{C}) = \mathcal{L}(\ell_2^p),$$

Example II

Let $p \in (1, \infty)$ and let q be its Hölder conjugate (i.e. $\frac{1}{p} + \frac{1}{q} = 1$).
For $d \in \mathbb{Z}_{\geq 1}$ we set $\ell_d^p = \ell^p(\{1, \ldots, d\})$.

Example

Let $A = \mathbb{C} = \mathcal{L}(\ell_1^p)$.

- $\mathsf{X} = \ell_d^p = \mathcal{L}(\ell_1^p, \ell_d^p)$.
- $\mathsf{Y} = \ell_d^q = \mathcal{L}(\ell_d^p, \ell_1^p)$.

Then, (X, Y) is C*-like module satisfying ❶ through ❻.

Cuntz-Pimsner algebras

For $d \in \mathbb{Z}_{\geq 2}$, the Cuntz-Pimnser construction can be carried (with minor modifications) for the C*-like module (ℓ_d^p, ℓ_d^q) over $\mathbb{C}$. This yields an L^p-operator algebra that we denote by $\mathcal{O}^p(\ell_d^p, \ell_d^q)$.

Theorem (D. 2021)

For $d \in \mathbb{Z}_{\geq 2}, \mathcal{O}^p(\ell_d^p, \ell_d^q)$ is isometrically isomorphic to $\mathcal{O}_d^p$, the L^p analogue of the Cuntz algebra $\mathcal{O}_d$ introduced by N. C. Phillips back in 2012.

Question

For which other C*-like modules can we construct $\mathcal{O}^p(\mathsf{X}, \mathsf{Y})$?

Example III

Example

Let A be an L^p operator algebra and let $d \in \mathbb{Z}_{\geq 2}$.

- $\mathsf{X} = \ell_d^p \otimes_p A = \mathcal{L}(\ell_1^p, \ell_d^p) \otimes_p A$.
- $\mathsf{Y} = \ell_d^q \otimes_p A = \mathcal{L}(\ell_d^p, \ell_1^p) \otimes_p A$.

Then, (X, Y) is C*-like module satisfying ❶ to ❺.

We don't know yet whether condition ❻ holds in general for this case. We suspect it does as long as A has a c.a.i..

Conjecture

$$\mathcal{O}^p(\mathsf{X}, \mathsf{Y}) \cong \mathcal{O}_d \otimes_p A.$$

Questions?