Games on AF-algebras

Andrea Vaccaro

Université de Paris

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(joint with B. de Bondt, B. Velickovic, A. Vignati)

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Given a formula $\varphi(\bar{x}) \in \mathcal{L}_{\mathrm{C}^*}$, a C^*-algebra A and a tuple $\bar{a} \in A^n$, we say that $\varphi^A(\bar{a})$ is **true**, or that $(A, \bar{a})$ **satisfies** $\varphi(x)$, if $\varphi^A(\bar{a}) = 0$.

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The **theory** of a C^*-algebra is the set of sentences φ such that $\varphi^A = 0$.

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- A C^*-algebra is abelian iff it satisfies the formula $\sup_x \sup_y \|[x, y]\|$.
- A unital C^*-algebra A satisfies the sentence

$$\bigvee_{k \in \mathbb{N}} \sup_{y_1, \ldots, y_k} \inf_{x_{i,j}} \max\{\{x_{i,j}\}_{i,j \leq 2} \text{ are matrix units }, \|[x_{i,j}, y_h]\|\}.$$

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  if and only if it is M_{2^∞}-stable, that is $A \otimes M_{2^\infty} \cong A$.
- There are also formulas determining whether a separable unital C^*-algebra is real rank zero, $\mathcal{Z}$-stable, $\mathcal{O}_2$-stable, simple, nuclear, quasidiagonal, AF, UHF, with finite nuclear dimension, and much more.

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The relations $\equiv_\alpha$ can be thought as increasingly precise approximations of the relation $\cong$.

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Let A be a separable C^*-algebra. Then there exists $\alpha < \omega_1$ such that $A \equiv_\alpha B$ implies $A \cong B$, for every separable C^*-algebra B.

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We now focus on unital AF-algebras.

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There exist unital AF-algebras A, B such that $A \equiv_n B$ for every $n \in \mathbb{N}$, but $A \not\cong B$.

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Although abelian concrete examples are easy to find (Eagle-Vignati, 2015), there were not known concrete simple unital examples witnessing the theorem above.

AF-algebras with high Scott's Rank

Question

Are there unital simple AF-algebras of arbitrarily high Scott's Rank?

That is, given $\alpha < \omega_1$, are there unital simple AF-algebras A, B such that $A \equiv_\alpha B$ but $A \not\cong B$?

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Our Theorem says: **YES!**

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Fix $n \in \mathbb{N}$, let A, B be two unital C^*-algebras and let $A_0 \subseteq A$, $B_0 \subseteq B$ be two dense subsets. The game $PI_n((A, A_0), (B, B_0))$ has n rounds and it is played by two players as follows.

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After n rounds the game ends, and Player II wins if $C^*(1_A, a_1, \dots, a_n) \cong C^*(1_B, b_1, \dots, b_n)$, otherwise Player I wins.

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Let A, B be two unital C^*-algebras and let $A_0 \subseteq A$, $B_0 \subseteq B$ be two dense subsets, and set $\alpha_0 = \alpha$, a countable ordinal. The game $PI_\alpha((A, A_0), (B, B_0))$ is played as follows. In round $k \in \mathbb{N} \setminus \{0\}$ first check if $\alpha_{k-1} > 0$. If that is the case then

- *Player I either plays a couple (α_k, a_k) or a couple (α_k, b_k), where $\alpha_k < \alpha_{k-1}$, and $a_k \in A_0$ or $b_k \in B_0$, respectively.*
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The proof is an adaptation of what happens in discrete model theory. For instance let $\mathcal{L}_{\mathrm{Gr}}$ be the set of formulas for in the language of groups:

- $\sum_{k<n} m_k x_k = 0$, with $m_k \in \mathbb{Z}$, belongs to $\mathcal{L}_{\mathrm{Gr}}$,
- if $\varphi(\bar{x}) \in \mathcal{L}_{\mathrm{Gr}}$, then $\neg\varphi(\bar{x}) \in \mathcal{L}_{\mathrm{Gr}}$,
- if $\varphi_0(\bar{x}), \varphi_1(\bar{x}) \in \mathcal{L}_{\mathrm{Gr}}$, then $\varphi_0(\bar{x}) \wedge \varphi_1(\bar{x})$ and $\varphi_0(\bar{x}) \vee \varphi_1(\bar{x})$ are in $\mathcal{L}_{\mathrm{Gr}}$,
- if $\varphi(\bar{x}, y) \in \mathcal{L}_{\mathrm{Gr}}$, then both $\exists y \varphi(\bar{x}, y), \forall y \varphi(\bar{x}, y) \in \mathcal{L}_{\mathrm{Gr}}$,
- if $\{\varphi_n(\bar{x})\}_{n\in\mathbb{N}} \subseteq \mathcal{L}_{\mathrm{Gr}}$ then $\bigvee_{n\in\mathbb{N}} \varphi_n(\bar{x}), \bigwedge_{n\in\mathbb{N}} \varphi_n(\bar{x}) \in \mathcal{L}_{\mathrm{Gr}}$.

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The proof is an adaptation of what happens in discrete model theory. For instance let $\mathcal{L}_{\mathrm{Gr}}$ be the set of formulas for in the language of groups:

- $\sum_{k<n} m_k x_k = 0$, with $m_k \in \mathbb{Z}$, belongs to $\mathcal{L}_{\mathrm{Gr}}$,
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It is possible to define a notion of rank between formulas in $\mathcal{L}_{\mathrm{Gr}}$, of $\equiv_\alpha$, Scott's Rank for groups and of $\mathrm{PI}_\alpha(G, H)$.

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Theorem (Fraïssé 1955, Karp 1965)

Player II has a winning strategy for $PI_\alpha(G, H)$ if and only if $G \equiv_\alpha H$.

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- We start an auxiliary match of $\mathrm{PI}_{\omega\cdot n}(K_0(A), K_0(B))$, assuming that in the first m rounds Player I plays $\{[e^{(j)}_{1,1}]\}_{j\leq m}$.

- Player II answers according to her winning strategy with some $\{d_j\}_{j\le m} \subseteq K_0(B)$ producing an ordered-groups homomorphism

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- **Round k$<$ n:** is similar to Round 1, but also uses **Uniqueness**.
- **Round n:** We end up with some injective map $\Phi_n\colon A_{r_n} \to B_{s_n}$ (or vice-versa) with $\Phi_n(a_i) = b_i$ for every $i < n$, hence $C^*(a_i) \cong C^*(b_i)$.

Reduce further to ordinals

Given an ordinal $\alpha < \omega_1$, consider the simple unital dimension group

$$\mathbb{G}_\alpha := (C(\alpha + 1, \mathbb{Q}), \ll, 1)$$

of continuous functions from α with the order topology (which makes it a 0-dimensional space), into $\mathbb{Q}$ with the discrete topology, where $f \ll g$ if $f(\beta) < g(\beta)$ for all $\beta \leq \alpha$.

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Theorem

Suppose that $\alpha + 1 \equiv_{\omega \cdot \theta} \beta + 1$ *as linear orders. Then* $\mathbb{G}_\alpha \equiv_\theta \mathbb{G}_\beta$ *as unital dimension groups.*

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Let $\{\epsilon_\alpha\}_{\alpha<\omega_1}$ be the (increasing) enumeration of all countable ordinals ϵ such that $\omega^\epsilon = \epsilon$.

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Furthermore, one can compute the Bratteli diagrams of all A_α by analyzing each $\mathbb{G}_{\epsilon_\alpha}$.

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