Connected Stable Rank of C*-algebras

(Joint work with Prahlad Vaidyanathan)

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Münster

Anshu
Indian Institute of Science Education and Research Bhopal

Overview

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- Properties and Examples

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- Estimates for $C(X)$-algebras when X has finite covering dimension

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- Rokhlin Property of a group action
- Estimates for Crossed product C^*-algebras when the group action has the Rokhlin property

Part I: Connected Stable Rank and K-theory

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- The connected stable rank is homotopy invariant, that is, any two homotopically equivalent C^*-algebras have the same rank.

Unimodular Vectors

Given a unital C^*-algebra A. For each $n \in \mathbb{N}$, we define

$$Lg_n(A) := \left\{ (a_i) \in A^n \text{ such that } \exists\ (b_i) \in A^n \text{ and } \sum_{i=1}^{n} b_i a_i = 1_A \right\}.$$

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Note that $e_i \in Lg_n(A)$ for any $1 \leq i \leq n$.

Connected Stable Rank

$GL_m(A) = \{ T \in M_m(A) : T \text{ is invertible } \}$.

Denote by $GL_m^0(A)$, the connected component of I_m in $GL_m(A)$ which is a normal subgroup of $GL_m(A)$.

$GL_m^0(A)$ acts on $Lg_m(A)$ by usual matrix multiplication.

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Ranks for non-unital C^*-algebras

If A is a non-unital C^*-algebra then the connected stable rank of A is defined as the connected stable rank of its minimal unitization, $\tilde{A}$, of A.

From now on, we assume that all C^*-algebras are unital.

Some interesting facts

- If $csr(A) = 1$, then A is stably finite.

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- If $csr(A) = 1$, $K_1(A) = 0$.

Relationship with K_1-group

[Rieffel, 1982]

Let A be a C^*-algebra. If $n \geq csr(A)$, then the natural maps

$$\frac{GL_{n-1}(A)}{GL^0_{n-1}(A)} \xrightarrow{\Phi_{n-1}} \frac{GL_n(A)}{GL^0_n(A)}$$

$$\bar{a} \mapsto \overline{\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}}$$

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$$K_1(A) \cong \frac{GL_{n-1}(A)}{GL^0_{n-1}(A)}.$$

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  In particular, if $csr(A) = 1$, $csr(M_n(A)) = 1$.
- For an extension $0 \longrightarrow J \longrightarrow A \longrightarrow B \longrightarrow 0$ of C^*-algebras,

$$csr(A) \leq max\{csr(J), csr(B)\}.$$

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- $csr(B(\mathcal{H})) = \infty$ if $\mathcal{H}$ is an infinite dimensional Hilbert space.
- $csr(C(X)) \leq \left\lceil \frac{dim(X)}{2} \right\rceil + 1$.

Part II: $C(X)$-algebras

$C(X)$-algebra

Definition

For a compact Hausdorff space X, a C^*-algebra A is said to be a $C(X)$-algebra if there is a unital $*$-homomorphism

$$\phi\colon C(X) \to Z(A)$$

where $Z(A)$ is the center of the C^*-algebra A.

We will assume, throughout the talk, that X is a compact Hausdorff space.

A $C(X)$-algebra as a bundle

- Let $x \in X$ be a fixed point, $I_x := \{f \in C(X) : f(x) = 0\}$ is an ideal in $C(X)$.

$$A(x) := A/I_x A \text{ is again a } \mathrm{C}^*\text{-algebra.}$$

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- For each $a \in A$, we get the natural quotient map

$$A \longrightarrow A(x), \ a \mapsto a(x).$$

and we define the map

$$X \to \mathbb{R}^+, \ x \mapsto \|a(x)\|.$$

It turns out the above map is always upper semicontinuous.

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- $csr(A)$ must depend on $dim(X)$, $csr(A(x))$ and "some" numbers which again depend on $dim(X)$.

Estimates for $C(X)$-algebras

Recall that for a given $C(X)$-algebra A, the fibres are defined as

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where $I_x = \{f \in C(X) : f(x) = 0\}$.

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Theorem (—— , Vaidyanathan)

Let A be a $C(X)$-algebra where X is a topological space of dimension zero. Then

$$csr(A) \leq \sup\{csr(A(x)) : x \in X\}.$$

Estimates for $C(X)$-algebras

Theorem (—— , Vaidyanathan)

Let A be a $C(X)$-algebra where X is an n-dimensional compact metric space. Then

$$csr(A) \leq \sup\{csr(\mathbb{T}^n A(x)) : x \in X\}.$$

Part III: Crossed Product C^*-algebras

Crossed product C^*-algebra

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- We can associate a C^*-algebra to the above system called the crossed product C^*-algebra and is denoted by $A \rtimes_\alpha G$.

- To each $g \in G$, associate the function $u_g \in C(G, A, \alpha)$ where $u_g(h) = 1$ iff $h = g$. Then $A \rtimes_\alpha G$ is the universal C^*-algebra generated by A and u_g subject to the relation $u_g a u_g^* = \alpha_g(a)$ for all $g \in G$.

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- Can we put some restrictions on the action α to get "nice" bounds?

Result

Definition

The topological stable rank of a C^*-algebra A, $tsr(A)$, is the least number $n \geq 1$ such that $Lg_n(A)$ is dense in A^n.

$$csr(A) \leq tsr(A) + 1.$$

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$$csr(A) \leq tsr(A) + 1.$$

Theorem (—— , Vaidyanathan)

Let G be a non-trivial group and let A be a C^*-algebra with topological stable rank one. Then

$$csr(A \rtimes G) \leq |G|.$$

Rokhlin property

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- $\sum_{g\in G} e_g = 1$
- $\alpha_g(e_h) = e_{gh}$ for all $g, h \in G$
- $\|e_g a - a e_g\| < \epsilon$ for all $g \in G, a \in F$.

Examples

- (A, G, id), the trivial action does not have Rokhlin property since $id_g = Id_A$ for all $g \in G$ then

$$id_{g^{-1}}(e_g) = e_g \neq e_{g_0} = e_{g^{-1}g}$$

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- Let B be any C^*-algebra. Then the left translation action

$$\alpha \colon G \longrightarrow Aut(C(G, B)) \text{ given by}$$

$$s \mapsto \alpha_s \text{ where } \alpha_s(f)(x) = f(s^{-1}x) \text{ for } f \in C(G, B).$$

For a given finite set $F \subset C(G, B)$ and a given $\epsilon > 0$, the following set of mutually orthogonal projections

$$e_g(h) = \begin{cases} 1_B, & \text{if } g = h \\ 0, & \text{otherwise.} \end{cases}$$

Theorem (——, Vaidyanathan)

Let $\alpha : G \to Aut(A)$ be an action of a finite group G on a separable, unital C^*-algebra A with the Rokhlin property. Then

- $$csr(A \rtimes_\alpha G) \leq \left\lceil \frac{csr(A) - 1}{|G|} \right\rceil + 1.$$

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- *In particular, if $csr(A) = 1$, then the same is true for $A \rtimes_\alpha G$.*

Application

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Now,

$$C(G, A) \rtimes_{\mathrm{lt}} G \cong M_n(A).$$

$$\text{Hence, } csr(M_n(A)) \leq \left\lceil \frac{csr(A) - 1}{n} \right\rceil + 1.$$

Blackadar's Example

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  - $$csr(A \rtimes_\alpha \mathbb{Z}_2) = 2.$$

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  - But $csr(A) = 1$.
- This crossed product does not satisfy our estimate. So we conclude that this bound will not work for more general actions.
- It shows that the bound

$$csr(A \rtimes_\alpha G) \leq |G|$$

is optimal.

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Thank You!