AF-algebras and Rational Homotopy Theory

(Based on joint work with Dr. Prahlad Vaidyanathan)

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Contents

The unitary and the quasi-unitary group

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- If B is a unital C*-algebra, we write $\mathcal{U}(B)$ for the group of unitaries in B.
- For a C*-algebra A, the map $\widehat{\mathcal{U}}(A) \to \mathcal{U}(A^+)$ given by $u \mapsto 1 - u$, is an isomorphism in case A is unital.

Homology theory on C*-algebras

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Definition

For any C*-algebra A and any $k \geq 0$, we set

$$G_k(A) := \pi_k(\widehat{\mathcal{U}}(A))$$

Then

- For each k, G_k is a homotopy invariant functor from the category of C*-algebras to the category of groups (abelian for $k \geq 1$).
- For $k \geq 0$, G_k is a continuous homology theory on the category of C*-algebras.

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We wish to *understand/calculate* this functor better.

Example: The Complex numbers

- $\mathcal{U}(\mathbb{C}) = S^1$

$$G_m(\mathbb{C}) \cong \pi_m(\mathcal{U}(\mathbb{C})) = \begin{cases} \mathbb{Z} & \text{if } m = 1 \\ 0 & \text{if } m \neq 1 \end{cases}$$

- However, for $n > 1$, $G_m(M_n(\mathbb{C})) \cong \pi_m(\mathcal{U}_n(\mathbb{C}))$ can be very complicated, and typically has torsion. For instance

$$G_6(M_2(\mathbb{C})) \cong \pi_6(\mathcal{U}_2(\mathbb{C})) = \mathbb{Z}_{12}$$

- By Bott periodicity, if $m > 1$ and $n \geq \frac{m+1}{2}$

$$G_m(M_n(\mathbb{C})) \cong \pi_m(\mathcal{U}_n(\mathbb{C})) = \begin{cases} 0 & \text{if } m \text{ even} \\ \mathbb{Z} & \text{if } m \text{ odd} \end{cases}$$

K-stability

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Let A be a C*-algebra and $j \geq 2$. Define $\iota_j : M_{j-1}(A) \to M_j(A)$ to be the natural inclusion map

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$$

Definition (Thomsen, 1991)

A C*-algebra A is said to be *K-stable* if

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Note:
$\mathbb{C}$ is not K-stable. In fact, any finite dimensional C*-algebra is not K-stable.

Example: The UHF-algebra of type 2^∞

$A = M_{2^\infty}$ is an inductive limit of

$$\mathbb{C} \to M_2(\mathbb{C}) \to M_4(\mathbb{C}) \to M_8(\mathbb{C}) \to \dots$$

Where the connecting maps are

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The map $\iota_2 : \mathcal{U}(A) \to \mathcal{U}_2(A)$ is then a homotopy equivalence because

$$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \sim_h \begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix}$$

in $\mathcal{U}_2(B)$, for any unital C^*-algebra B.

Other examples of $\mathcal{K}$-stable C*-algebras

- [Rieffel, 1987] Non-commutative tori A_θ, with θ irrational.

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Proposition (Thomsen, 1991)

For any C*-algebra A, $G_k(\mathcal{K} \otimes A)$ is naturally isomorphic to $K_{k+1}(A)$, $k \geq 0$.

Hence, to say that a C*-algebra is K-stable, is to say that

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Thus, for $A = M_{2^\infty}$, for each $n \in \mathbb{N}$

$$G_k(M_n(A)) \cong K_{k+1}(A) \cong \begin{cases} \mathbb{Z}\left[\frac{1}{2}\right] & \text{if } k \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$$

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Question?

What can be said about the K-stability of AF-algebras?

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Problem...

Even for a finite dimensional C*-algebra, computation of G_k can be very complicated.

Rational Homotopy Theory

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- It is primarily to remedy this difficulty, that topologists introduced *Rational Homotopy theory*.
- Homotopy theory is the study of spaces with homotopy equivalence. In rational homotopy theory one simplifies these invariants. Instead of $H_n(\cdot)$ and $\pi_n(\cdot)$, we consider $H_n(\cdot;\mathbb{Q})$ and $\pi_n(\cdot)\otimes\mathbb{Q}$.

Definition

For any C*-algebra A and any $m \geq 1$, we set

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Example: Revisiting the Complex numbers

For $n \in \mathbb{N}$

$$\mathcal{U}_n(\mathbb{C}) \simeq_{\mathbb{Q}} S^1 \times S^3 \ldots \times S^{2n-1}$$

$$F_m(M_n(\mathbb{C})) = \pi_m(\mathcal{U}_n(\mathbb{C})) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & \text{if } 1 \leq m \leq 2n-1, m \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

Finite dimensional C*-algebra

Given a tuple $\overline{p} = (p_1, p_2, \dots, p_n)$ of positive integers, we consider the finite dimensional C*-algebra associated to it by

$$M(\overline{p}) := M_{p_1}(\mathbb{C}) \oplus M_{p_2}(\mathbb{C}) \oplus \dots \oplus M_{p_n}(\mathbb{C})$$

By additivity of the functor F_m, we have

$$F_m(M(\overline{p})) = \bigoplus_{j=1}^{n} \mathbb{Q}^{d(m,j)} \text{ where}$$

$$d(m,j) = \begin{cases} 1 & \text{if } 0 \leq m \leq 2p_j - 1, m \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

Multiplicity of a $*$-homomorphism

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- Let $\varphi : M(\overline{p^1}) \to M(\overline{p^2})$ be a $*$-homomorphism between two finite dimensional C*-algebras. For $1 \leq j \leq n_1$ and $1 \leq i \leq n_2$, define $\varphi_{i,j} : M(p^1_j) \to M(p^2_i)$ to be the map given by

$$M_{p^1_j}(\mathbb{C}) \hookrightarrow M(\overline{p^1}) \xrightarrow{\varphi} M(\overline{p^2}) \twoheadrightarrow M_{p^2_i}(\mathbb{C})$$

  Then the multiplicity of $\varphi_{i,j}$ is

$$\ell_{i,j} := \frac{Tr(\varphi_{i,j}(e))}{Tr(e)}$$

  where e is any non-zero projection in $M(p^1_j)$. Note that this formula is independent of the choice of e.

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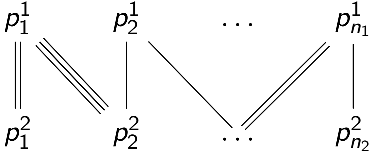
- We write

$$\Phi := [\ell_{i,j}] \in M_{n_2,n_1}(\mathbb{Z}^+)$$

  for the multiplicity matrix associated to φ.

Multiplicity of a $*$-homomorphism

- Furthermore, the map φ is unitarily equivalent to a map $\psi : A_1 \to A_2$ which may be represented by a diagram $\mathcal{D}(A_1, A_2, \varphi)$ as



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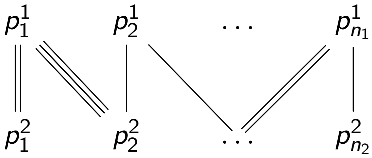

  where the number of lines connecting p_j^1 to p_i^2 is $\ell_{i,j}$.
- Since the unitary group of a finite dimensional C^*-algebra is connected, two unitarily equivalent $*$-homomorphisms are homotopic, and hence induce same maps at the level of $G_k(\cdot)$ and $F_m(\cdot)$.

Computing $F_m(\varphi)$

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Theorem (Apurva Seth, Prahlad Vaidyanathan, 2020)

For $k = 1, 2$, let $\overline{p^k} = (p_1^k, p_2^k, \ldots, p_{n_k}^k)$ be two tuples of positive integers. Given a $*$-homomorphism $\varphi : M(\overline{p^1}) \to M(\overline{p^2})$ and $m \in \mathbb{N}$,

$$F_m(\varphi) : F_m(M(\overline{p^1})) \to F_m(M(\overline{p^2}))$$

is given by multiplication by its multiplicity matrix Φ.

An illustration

To illustrate the above theorem, we give an example. Let

$$A_1 = \mathbb{C} \oplus M_2(\mathbb{C}) \oplus M_3(\mathbb{C}), \text{ and } A_2 = \mathbb{C} \oplus M_3(\mathbb{C}) \oplus M_5(\mathbb{C}) \oplus M_8(\mathbb{C})$$

and $\varphi : A_1 \to A_2$ is given by

$$\varphi(x, y, z) := \left(x, \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, \begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & z \end{pmatrix} \begin{pmatrix} y & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & z \end{pmatrix} \right)$$

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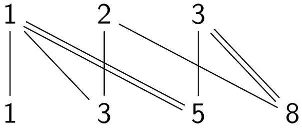
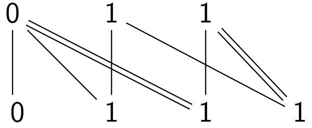


And the multiplicity matrix is

$$\Phi = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

Then for $m = 3$, we get a corresponding diagram of $\mathbb{Q}$-vector spaces $\mathcal{D}_3(A_1, A_2, \varphi)$



so that $F_3(\varphi) : \mathbb{Q}^2 \to \mathbb{Q}^3$ is the map $(b, c) \mapsto (b, c, b + 2c)$.

Rational K-stability

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Definition

A C*-algebra A is said to be *rationally K-stable* if

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is an isomorphism for all $m \geq 1$ and all $j \geq 2$.

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Note:

- For a rationally K-stable C*-algebra A, for $m \geq 1$, $F_m(A) \cong K_{m+1}(A) \otimes \mathbb{Q}$.
- The notion of rational K-stability has been studied by Farjoun and Schochet where it was termed as *rational Bott-stability*.

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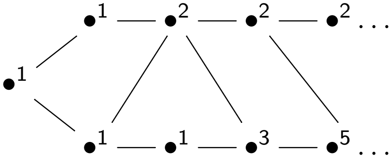
Consider the AF-algebra, A given by the following Bratteli diagram



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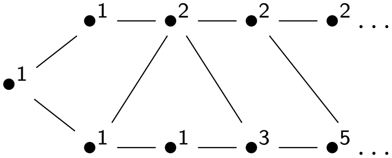

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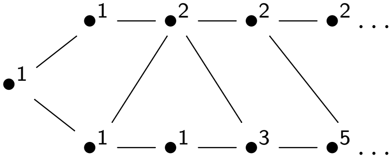

- Then, A has an ideal I such that $A/I \cong M_2(\mathbb{C})$.
- For $n > 1$, consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I & \longrightarrow & A & \longrightarrow & M_2(\mathbb{C}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_n(I) & \longrightarrow & M_n(A) & \longrightarrow & M_{2n}(\mathbb{C}) & \longrightarrow & 0
\end{array}$$

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$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_m(I) & \longrightarrow & F_m(A) & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle F_m(\iota_I)} & & \downarrow{\scriptstyle F_m(\iota_A)} & & \downarrow{\scriptstyle F_m(\iota_Q)} & & \\ 0 & \longrightarrow & F_m(M_n(I)) & \longrightarrow & F_m(M_n(A)) & \longrightarrow & \mathbb{Q} & \longrightarrow & 0 \end{array}$$

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\end{array}$$

- Since $F_m(\iota_Q)$ is the zero map (and hence not surjective), it follows that $F_m(\iota_A)$ cannot be surjective. Thus, A can't be rationally K-stable.

Main Result

Theorem (Apurva Seth, Prahlad Vaidyanathan, 2020)

For an AF-algebra A, the following are equivalent:

1. A is K-stable.
2. A is rationally K-stable.
3. A has no non-zero finite dimensional representations.
4. For each $m \in \mathbb{N}$, there is a generating nest $\{A_{m,p} : p \in \mathbb{N}\}$ of finite dimensional C*-algebras such that

$$\min \dim(A_{m,p}) \geq m$$

for all $p \in \mathbb{N}$.

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5. Analogous to K-stability, we defined rational K-stability and found an obstruction to this property and hence an obstruction to K-stability.
6. We proved that, that particular obstruction was the only obstruction to K-stability and both notion coincide for class of AF-algebras.

Thank You!