Bernoulli actions of type III and their von Neumann algebras

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August 12, 2021

Joint work with Stefaan Vaes

*FWO fellowship, research grant 1139719N

Nonsingular group actions

Fix a countable group Γ acting on a probability space (X, μ).

Definition

We say that the action $\Gamma \curvearrowright (X, \mu)$ is nonsingular if

$$\mu(g \cdot U) = 0 \Leftrightarrow \mu(U) = 0.$$

The nonsingularity condition is needed to obtain a well-defined action

$$\Gamma \curvearrowright L^\infty(X, \mu),\ (g \cdot f)(x) = f(g^{-1}x).$$

A prominent class of von Neumann algebras arises from nonsingular group actions.

Bernoulli actions

A very interesting class of group actions are the Bernoulli actions.

Definition

Let Γ be a countable group and X_0 a measurable space with probability measures $\{\mu_g\}_{g\in\Gamma}$. We consider the product space

$$(X, \mu) = \prod_{g\in\Gamma} (X_0, \mu_g)$$

equipped with the **Bernoulli action** $\Gamma \curvearrowright (X, \mu)$ given by $(g \cdot x)_h = x_{g^{-1}h}$.

Definition of the group measure space construction

Let $\Gamma \curvearrowright (X, \mu)$ be a nonsingular action. Recall this induces an action $\Gamma \curvearrowright L^\infty(X, \mu)$ given by $\sigma_g(f)(x) = f(g^{-1}x)$.

Definition

The **group measure space construction** of the action $\Gamma \curvearrowright (X, \mu)$, denoted by $L^\infty(X, \mu) \rtimes \Gamma$, is the von Neumann algebra generated by $L^\infty(X, \mu)$ and $L(\Gamma)$ acting on the Hilbert space $\mathcal{H} = L^2(X, \mu) \otimes \ell^2(\Gamma)$ with relations

$$u_g^* = u_g^{-1} \text{ and } u_g f u_g^* = \sigma_g(f).$$

This construction is also called the **crossed product**.

Properties of $L^\infty(X,\mu) \rtimes \Gamma$

Remarks. Denote $M = L^\infty(X,\mu) \rtimes \Gamma$.

1. Every $x \in M$ can be written as

$$x = \sum_{g \in \Gamma} x_g u_g$$

   with $x_g \in L^\infty(X,\mu)$.
2. When $\Gamma \curvearrowright (X,\mu)$ is essentially free and ergodic, then M is a factor.
3. The measure μ determines a faithful normal state on M given by

$$\varphi : M \to \mathbb{C} : \sum_{g \in \Gamma} x_g u_g \mapsto \int_X x_e d\mu.$$

4. When $\Gamma \curvearrowright (X,\mu)$ is measure preserving, φ is a tracial state.
5. Orbit equivalent actions give isomorphic crossed products.

Type classification

Given an essentially free, ergodic, nonsingular action $\Gamma \curvearrowright (X, \mu)$. The type of the von Neumann algebra $L^\infty(X, \mu) \rtimes \Gamma$ is,

Proposition

- I *if Γ acts transitively, i.e. $\exists x \in X : X = \Gamma x$ mod μ.*

If Γ acts non-transitively (μ has no atoms), then the type is,

- II_1 *if there exists a Γ-invariant probability measure, equivalent with μ,*
- II_∞ *if there exists a Γ-invariant σ-finite measure, equivalent with μ,*
- III *if there is no Γ-invariant measure equivalent with μ.*

Background on Tomita-Takesaki theory

Let M be a von Neumann algebra equipped with a faithful normal state φ. Very deep analysis (Tomita-Takesaki theory) gives two canonical modular operators J_φ and Δ_φ on $L^2(M, \varphi)$ satisfying

- $J_\varphi M J_\varphi = M'$,
- $\sigma^\varphi : \mathbb{R} \to \mathrm{Aut}(M), \sigma_t^\varphi(x) = \Delta_\varphi^{it} x \Delta_\varphi^{-it}$, a one-parameter group.

With the action $\mathbb{R} \curvearrowright^{\sigma^\varphi} M$ comes the crossed product $M \rtimes_{\sigma^\varphi} \mathbb{R}$, that we denote by $c_\varphi(M)$. This is called the continuous core of M.

Tomita-Takesaki theory for $L^\infty(X,\mu) \rtimes \Gamma$

Let $\Gamma \curvearrowright (X,\mu)$ be any nonsingular action. **Define**

$$\omega : \Gamma \times X \mapsto \mathbb{R}_*^+,\ \omega(g,x) = \frac{d\mu(g\cdot\ )}{d\mu}(x).$$

This is a very useful tool! For example, $\Gamma \curvearrowright (X,\mu)$ is measure preserving if and only if $\forall g \in \Gamma : \omega(g,\cdot) = 1$. An easy exercise shows that

$$\Delta_{\varphi_\mu} : L^2(X,\mu) \otimes \ell^2(\Gamma) \to L^2(X,\mu) \otimes \ell^2(\Gamma) :$$
$$f_g \otimes \delta_g = \omega(g,\cdot) f_g \otimes \delta_g.$$

How to distinguish type III von Neumann algebras?

Use invariants introduced for general von Neumann algebras and compare!

1. The type classification: **I**, **II**, **III**,
2. The subtype $\mathbf{III}_\lambda$ with $\lambda \in [0, 1]$,
3. For full factors, Connes' τ-invariant: the weakest topology on $\mathbb{R}$, making the one parameter group $\mathbb{R} \to \mathsf{Out}(M) : t \mapsto \sigma_t^\varphi$ continuous.

There are still many factors that can't be distinguished by these invariants!

- (Shlyakthenko 2002): pair of free Araki-woods factors,
- (Houdayer-Shlyakhtenko-Vaes 2017): large family of free Araki-woods factors.

Beyond the standard invariants for factors

Concept of distinguishing factors with the same modular invariants.

Given a self-adjoint (unbounded) operator A, we denote by $\mathcal{C}(A)$ the measure class of the spectral measure of A, i.e. for a subset $U \subset \mathbb{R}$

$$U \in \mathcal{C}(A) \Leftrightarrow \mathbb{1}_U(A) = 0.$$

Let M, M' be any two factors with faithful normal states φ, φ'.

- Whenever $M \cong M'$, the subtype III$_\lambda$, $\lambda \in [0,1]$ is the same. For our purposes, this means that

$$\sigma(\Delta_\varphi) = \sigma(\Delta_{\varphi'}).$$

- Whenever there exists a state preserving isomorphism $(M, \varphi) \cong (M', \varphi')$, we get a unitary U such that

$$\Delta_\varphi = U \Delta_{\varphi'} U^*,$$

  in particular,

$$\mathcal{C}(\Delta_\varphi) = \mathcal{C}(\Delta_{\varphi'}).$$

Main theorem

For the free group $\mathbb{F}_n$, $n \geq 3$, write $\mathbb{F}_n = \mathbb{F}_{n-1} * \mathbb{Z}$.

Theorem (Vaes-V)

Fix $n \geq 3$. Let $\mathbb{F}_n \curvearrowright (X, \mu)$ and $\mathbb{F}_n \curvearrowright (X', \mu')$ be any two nonsingular Bernoulli actions, with measures $\{\mu_g\}_{g \in \mathbb{F}_n}$ and $\{\mu'_g\}_{g \in \mathbb{F}_n}$ depending only on the last letter of g w.r.t. $\mathbb{Z}$. Suppose that

$$L^\infty(X, \mu) \rtimes \mathbb{F}_n \cong L^\infty(X', \mu') \rtimes \mathbb{F}_n,$$

then

$$\mathcal{C}(\Delta_\mu) = \mathcal{C}(\Delta_{\mu'}).$$

Remark. Note that $\mathbb{F}_{n-1}$ preserves μ/ all non-invariance comes from the group $\mathbb{Z}$.

Concrete examples of type III Bernoulli actions

Vaes and Wahl (2017): Concrete examples of type III Bernoulli actions $\mathbb{F}_n = \mathbb{F}_{n-1} * \mathbb{Z} \curvearrowright (X, \mu) = \prod_{g \in \Gamma} (X_0, \mu_g)$ given by

1. The base space X_0: arbitrary.
2. Probability measures $\{\mu_g\}_{g \in \Gamma}$: take two equivalent probability measures μ_0, μ_1 on X_0 and define

$$\mu_g = \begin{cases} \mu_1 & \text{if the last letter of } g \text{ belongs to } \mathbb{N} \subset \mathbb{Z}, \\ \mu_0 & \text{otherwise.} \end{cases}$$

Consequence of the main theorem

Corollary (Vaes-V)

There exists uncountable many non isomorphic group measure space constructions of type III_1 and τ-invariant.

This adds to the previous list:

- (Shlyakthenko 2002): pair of free Araki-woods factors,
- (Houdayer-Shlyakhtenko-Vaes 2017): large family of free Araki-woods factors.

Thanks