Topological Insulators

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The Hamiltonian Operator H

What are Topological Insulators?

The Hamiltonian operator acts on the hilbert space $l^2(\mathbb{Z}^d, \mathbb{C}^N)$. The relevant C^* algebra is $C^*(\mathbb{Z}^d, \mathbb{M}_n(\mathbb{C})) \cong C(\mathbb{T}^d, \mathbb{M}_n\mathbb{C})$.

- $H \in C(\mathbb{T}^d, \mathbb{M}_n(\mathbb{C}))$
- $H = H^*$
- H insulator $\iff$ H invertible

Introducing K-theory

- H gives a spectral projection $P \in C(\mathbb{T}^d, \mathbb{M}_{\mathbb{N}}(\mathbb{C}))$
- Homotopy class of projections defines a non trivial $K-$ theory class $K_0(C(\mathbb{T}^d)) = K^0(\mathbb{T}^d)$

Bulk and Half space

- Replace $\mathbb{Z}^d$ by $\mathbb{N} \times \mathbb{Z}^{d-1}$ and describe H on $l^2(\mathbb{N} \times \mathbb{Z}^{d-1})$
- For $d = 1$, $\hat{H} \in \tau$, the Toeplitz C^* algebra, where $\hat{H}$ represents the half space Hamiltonian
- For $d \geq 2$, $\hat{H} \in \tau \otimes C^*(\mathbb{Z}^{d-1}) \otimes \mathbb{M}_N(\mathbb{C}) = \mathbb{M}_N(C(\mathbb{T}^{d-1}, \tau))$

- $C(\mathbb{T}^{d-1}, \tau) \twoheadrightarrow C(\mathbb{T}^d)$, $\hat{H} \longmapsto H$
- if spectral projection does not lift, then the half space system conducts. $0 \in \sigma(\hat{H})$

Further Work

- Joint work with Ralf Meyer
- What is the explicit formula for the Hamiltonian that generates the K-theory of $C^*(\mathbb{Z}^d)$?
- Continuous map from $\mathbb{T}^d \longrightarrow \mathbb{S}^d$
- pullback $C(\mathbb{S}^d) \longrightarrow C(\mathbb{T}^d)$
- $K_*(C(\mathbb{T}^d)) \longrightarrow K_*(C(\mathbb{S}^d))$

$$C(\mathbb{T}^{d-1}, \mathbb{K}(\ell^2(\mathbb{N}))) \rightarrowtail C(\mathbb{T}^{d-1}, \mathcal{T}) \twoheadrightarrow C(\mathbb{T}^d)$$

↑ boundary ↑ half space ↑ bulk

Bott periodicity gives the reverse index map (exp map)

$$K_0(C(\mathbb{T}^d)) \xrightarrow{\exp} K_1(C(\mathbb{T}^{d-1}, \mathbb{K}(\ell^2(\mathbb{N}))) \cong K_1(C(\mathbb{T}^{d-1}))$$

kernel given by image of:

$$K_0(C(\mathbb{T}^{d-1}, \mathcal{T})) \rightarrow K_0(C(\mathbb{T}^d))$$

If $\hat{P}$ lifts P, then P is in the image of this map. $\exp(P) = 0$. If $\exp(P) \neq 0$ $\hat{H}$ not invertible.