The ideal intersection property for essential groupoid C*-algebras

Dan Ursu
Joint work with Matthew Kennedy, Se-Jin Kim, Xin Li, Sven Raum

University of Waterloo

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Disclaimers

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This talk will give the ideal convex combination of hand-wavy and rigorous for the question "what's an étale groupoid"?

Crossed products

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Intuition: contains $\{\sum_{\text{finite}} a_t \lambda_t \mid t \in G, a_t \in A\}$ as a dense subset, and

$$a\lambda_s b\lambda_t = a\lambda_s b\lambda_s^* \lambda_s \lambda_t = (a(s \cdot b))\lambda_{st}.$$

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The reduced crossed product $A \rtimes_r G$ is the unique norm completion such that there is a faithful conditional expectation $E : A \rtimes_r G \to A$ satisfying $E(\sum a_g \lambda_g) = a_e$.

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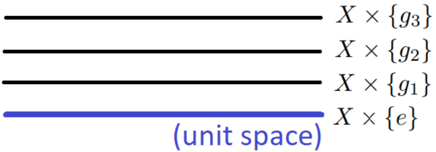

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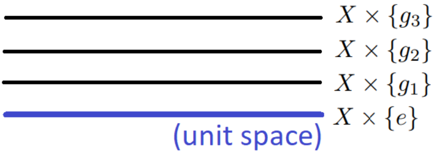

Under appropriate structure, $X \times G$ is a **transformation groupoid**, and $C_r^*(\mathcal{G}) = C(X) \rtimes_r G$.

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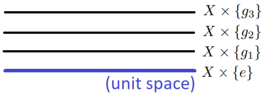

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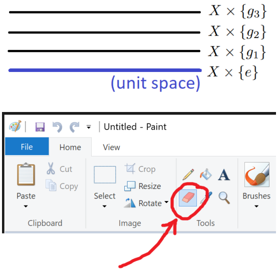

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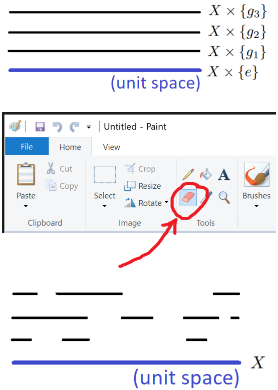
= general(ish) étale groupoid



group elements $g \subseteq \mathcal{G}$ acting by homeomorphisms

inverse semigroup elements $s \subseteq \mathcal{G}$ acting by partially defined homeomorphisms

What's a groupoid? Part 3.

In other words, a general étale groupoid $\mathcal{G}$ looks like $X \rtimes S$ for an inverse semigroup S acting by **partial homeomorphisms**. Can assume each $s \in S$ satisfies $s \subseteq \mathcal{G}$. (These are bisections).

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Similarly, $C_r^*(\mathcal{G}) = C(X) \rtimes_r S$ for an appropriate definition of crossed product. There is again a faithful conditional expectation

$$E(f\lambda_s) = \begin{cases} f & \text{if } s \subseteq X \\ 0 & \text{if } s \cap X = \emptyset \end{cases}$$

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   One direction. If $\mathcal{G}$ has no amenable confined subgroup in some (any) isotropy group, then $C_r^*(\mathcal{G})$ is simple.

The category of $\mathcal{G}$-C*-algebras

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Importantly, $C_r^*(\mathcal{G})$ is an object in the new category!

Injective envelopes

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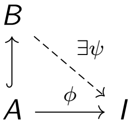

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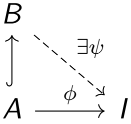

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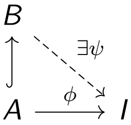

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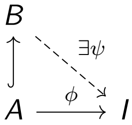

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Theorem (K-K-L-R-U, 2021)

In our new category of $\mathcal{G}$-C-algebras, there is an injective envelope of $C(X)$. Denote it by $C(\partial_F \mathcal{G})$, where $\partial_F \mathcal{G}$ is the Furstenberg boundary of $\mathcal{G}$.*

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Idea in all C*-simplicity arguments:
$C_r^*(\partial_F \mathcal{G} \rtimes \mathcal{G}) = C_r^*(\partial_F \mathcal{G} \rtimes S) = C(\partial_F \mathcal{G}) \rtimes S$ is a much nicer object to work with.

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4. *There is a unique pseudoexpectation* $E : C_r^*(\mathcal{G}) \to C(\partial_F \mathcal{G})$.

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Consider $\mathcal{G} = X \rtimes S$ and $x \in X$. Define

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Other corollaries (Powers averaging, C*-irreducible inclusions, etc...) follow from properties of $\partial_F \mathcal{G}$ but won't fit in 20 minutes.

- FIN -