Cyclic homology and non-commutative geometry in positive characteristic

Devarshi Mukherjee

joint with Guillermo Cortiñas and Ralf Meyer

[1]Mathematisches Institut
Universität Göttingen

YMC*A, WWU Münster, 2021

Goals of the program

- Develop non-commutative geometry in positive characteristic, and over $\mathbb{Z}_p$ and $\mathbb{Q}_p$;
- Find a non-commutative generalisation of the appropriate de Rham theory for affine varieties over $\mathbb{F}_p$.

Why is this interesting? - I

- Group actions on affine varieties over $\mathbb{F}_p \rightsquigarrow$ crossed product algebras;
- Graphs $\rightsquigarrow$ Leavitt path algebras;
- Carry over tools from rigid geometry to non-commutative geometry.

Why is this interesting? - II

- A fundamental invariant of a smooth manifold M is its de Rham cohomology $\mathrm{hdR}^*(M)$;
- These are the cohomology groups of the cochain complex

$$0 \to \mathrm{C}^\infty(M) \xrightarrow{d} \Omega^1(M) \to \cdots \xrightarrow{d} \Omega^n(M),$$

- Hochschild homology is an invariant of associative algebras that generalises differential forms. That is, $\mathrm{HH}_k(\mathrm{C}^\infty(M)) \cong \Omega^k(M)$,
- Periodic cyclic homology is a non-commutative generalisation of de Rham cohomology: $\mathrm{HP}_n(\mathrm{C}^\infty(M)) \cong \bigoplus_{j\in\mathbb{Z}} \mathrm{hdR}^{n+2j}(M),\ n = 0, 1.$

Why is this interesting? - II (continued)

- Periodic cyclic homology conveys important information about 'topological' algebras, through its link with (bivariant) K-theory;
- Want similar information about topological algebras over $\mathbb{Q}_p$ and $\mathbb{Z}_p$;
- In the commutative world, we want a similar link with de Rham cohomology as in the smooth manifold case, but for affine varieties over $\mathbb{F}_p$;
- What are the right topological algebras and invariants in this setting?

What goes wrong in positive characteristic?

- De Rham cohomology is homotopy invariant - so contractible spaces have no cohomology;
- The proof of this relies on integration of differential forms - this leads to denominators;
- But in positive characteristic, you are not allowed to divide;
- So (algebraic) de Rham cohomology is undesirable in positive characteristic.

The correct de Rham theory in positive characteristic

- Lift a smooth commutative $\mathbb{F}_p$-algebra A to a smooth commutative $\mathbb{Z}_p$-algebra R;
- Obvious things - take the de Rham cohomology of R or its p-adic completion $\widehat{R}$ - these fail;
- Instead, take something in between: $R \subseteq R^\dagger \subseteq \widehat{R}$. It consists of power series, whose coefficients converge somewhat rapidly;
- The de Rham cohomology of $\underline{R}^\dagger := R^\dagger \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is the right thing to consider - called rigid cohomology;
- Cortiñas-Cuntz-Meyer-Tamme, 2017: $\mathrm{HP}_n(\underline{R}^\dagger) \cong \bigoplus_{j \in \mathbb{Z}} \mathrm{hdR}^{n+2j}(R/pR)$, $n = 0, 1$

Towards local and analytic cyclic homology - I

- In the non-commutative world, it is not obvious what $R \subseteq R^\dagger \subseteq \widehat{R}$ should mean;
- We do this by making sense of convergent power series in non-commuting variables - uses bornological analysis;
- It is inspired by the definition of a smooth subalgebra $A^\infty \subseteq A$ of a C^*-algebra, which generalises $\mathrm{C}^\infty(M) \subseteq \mathrm{C}(M)$;
- We would like an invariant X that satisfies $X(R^\dagger) \cong X(\widehat{R})$;
- In the complex case, $X = \mathrm{HL}(=\mathrm{HA})$ does the job and provides a powerful invariant for C^*-algebras.

Towards local and analytic cyclic homology - II

- We additionally want our invariant to be independent of choices of liftings of an $\mathbb{F}_p$-algebra to torsion-free $\mathbb{Z}_p$-algebras;
- It is unclear that HP has such properties.

Theorem (Cortiñas-Meyer-Mukherjee)

We construct a functor

$$\mathrm{HA}\colon \{\mathbb{F}_p\text{-algebras}\} \longrightarrow \{\mathbb{Q}_p\text{-vector spaces}\}$$

that satisfies homotopy invariance, excision, Morita invariance, independence of choices of liftings, etc.

Some computations

- For smooth curves over $\mathbb{F}_p$, our theory recovers rigid cohomology;
- For Leavitt path algebras, we recover the computation of HP $\rightsquigarrow$ get something which only depends on the incidence matrix of the graph.

Summary

- Over $\mathbb{C}$, periodic cyclic homology generalises de Rham cohomology;
- Over $\mathbb{F}_p$, periodic cyclic homology of a $\mathbb{Z}_p$-algebra lift of the form $pR^\dagger \rightarrowtail R^\dagger \twoheadrightarrow A$ yields the rigid cohomology of A;
- We define a well-behaved invariant that is independent of choices of 'liftings' of A to torsion-free $\mathbb{Z}_p$-algebras;
- When A is the coordinate ring of a smooth curve over $\mathbb{F}_p$, our invariant agrees with rigid cohomology.