A torsion-free C^*-unique group

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Given a group Γ, there is a bijection between ideals in $C^*(\Gamma)$ and C^*-seminorms on $\mathbb{C}\Gamma$.
Analogously, there is a bijection between *C^*-norms* on $\mathbb{C}\Gamma$ and ideals $I \trianglelefteq C^*(\Gamma)$ such that $I \cap \mathbb{C}\Gamma = \{0\}$ is trivial.
A group Γ is said to be *C^*-unique* if $\mathbb{C}\Gamma$ admits a unique C^*-norm.

Example

If Γ is finite, then $\mathbb{C}\Gamma = C^*(\Gamma)$ and Γ is C^*-unique.
Moreover, any C^*-unique group is amenable.
On the other hand, e.g. $\mathbb{Z}$ is not C^*-unique.

Grigorchuk-Musat-Rørdam observed that any locally finite group is C^*-unique, and asked whether the converse holds.
In a workshop in 2019, Alekseev presented his work on C^*-uniqueness. Ozawa, who was attending the workshop, pointed out that the lamplighter group

$$\bigoplus_{\mathbb{Z}} \mathbb{Z}_2 \rtimes \mathbb{Z}$$

is C^*-unique (and not locally finite). Main ingredient is the fact that $\mathbb{Z} \curvearrowright \widehat{\bigoplus_{\mathbb{Z}} \mathbb{Z}_2}$ is a *topologically free* action.
Alekseev asked then whether there is a *torsion-free* C^*-unique group.

Given an integer $n \geq 2$, $\mathbb{Z}[1/n] = \{\frac{k}{n^l} : k \in \mathbb{Z}, l \in \mathbb{N}\}$.
Two integers $p, q \geq 2$ are said to be *multiplicatively independent* if there is no $r, s \in \mathbb{N}$ such that $p^r = q^s$.

Theorem (S.)

Let p and q be multiplicatively independent integers. Then $\mathbb{Z}[\frac{1}{pq}] \rtimes \mathbb{Z}^2$ is C^*-unique.

The main ingredient is a theorem of Furstenberg which says that, given p and q multiplicatively independent integers, if $B \subset S^1$ is an infinite closed set such that $B = \{z^p : z \in B\} = \{z^q : z \in B\}$ then $B = S^1$.
The measure-theoretical counterpart of this theorem is an open problem (Furstenberg conjecture): whether the only $\times p$-$\times q$-invariant ergodic probability measure on S^1 with infinite support is the Lebesgue measure.