Étale equivalence relations and C^*-algebras for iterated function systems

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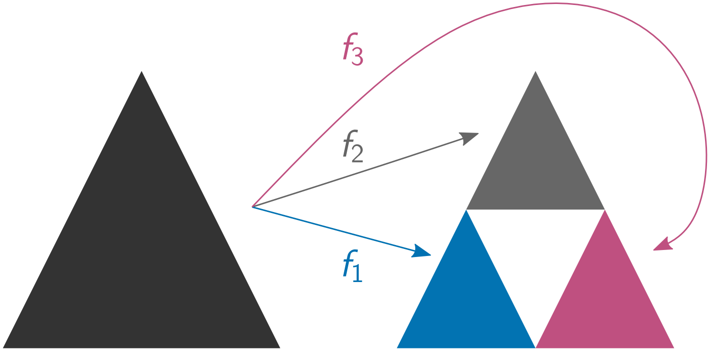
Outline

Iterated Function Systems

An **iterated function system (IFS)** is a finite collection of contractive functions on $\mathbb{R}^d$:

$$\mathcal{F} = \left\{ f_i : \mathbb{R}^d \to \mathbb{R}^d \ : \ ||f_i(x) - f_i(y)|| \leq \lambda_i ||x - y||, \ \lambda_i \in (0,1) \right\}_{i=1}^{m}.$$

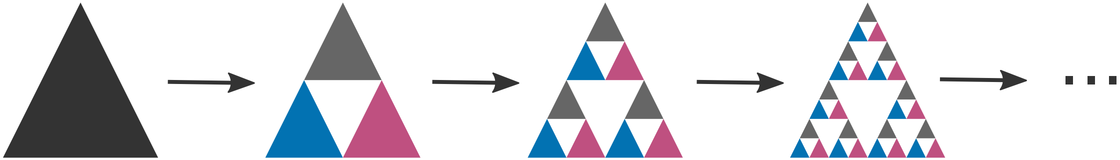


Hutchinson's Theorem [1981]

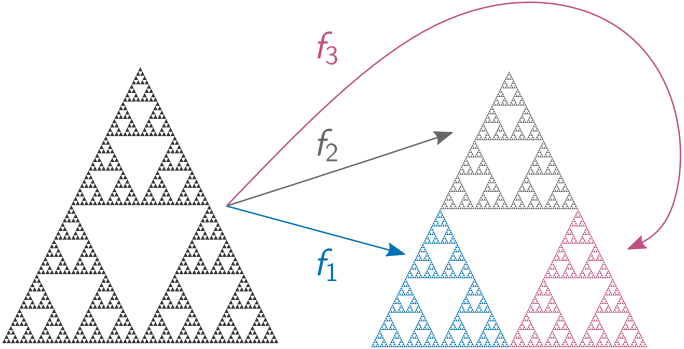
Let $\mathcal{F} = \{f_1, \ldots, f_m\}$ be an IFS. Then, there exists a unique, non-empty, compact subset $K \subseteq \mathbb{R}^d$ such that

$$K = \bigcup_{i=1}^{m} f_i(K).$$

This set K is called the **attractor** of the IFS.



The **Siérpinski Gasket** is the attractor of the IFS $\mathcal{F} = \{f_1, f_2, f_3\}$.



$$K = f_1(K) \cup f_2(K) \cup f_3(K)$$

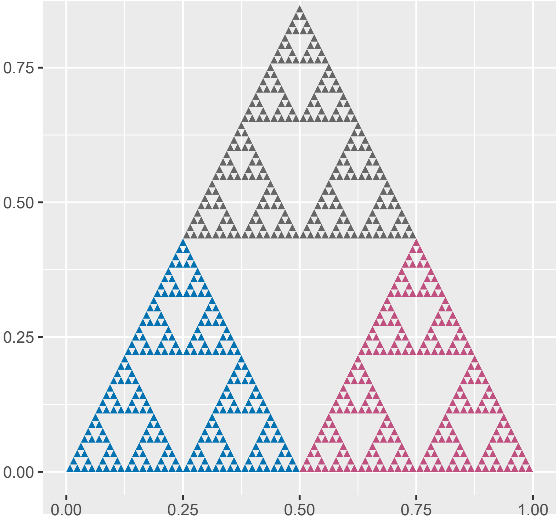
The Standard Siérpinski Gasket IFS

$f_1(x) = \frac{1}{2}x$

$f_2(x) = \frac{1}{2}x + \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$

$f_3(x) = \frac{1}{2}x + \begin{bmatrix} 1/4 \\ \sqrt{3}/4 \end{bmatrix}$

same scale + different shifts



C^*-algebras for Iterated Function Systems

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- Standard Siérpinski Gasket IFS does not satisfy this condition
- **Goal:** Provide a groupoid C^*-algebra for a broad class of IFS including the standard Siérpinski Gasket IFS

Approach:

1. Define equivalence relation $R \subseteq K \times K$
2. Construct an **étale topology** on R such that
   - $R^2 = \{((x, y_1), (y_2, z)) \in R \times R \mid y_1 = y_2\}$ closed
   - Product $((x, y), (y, z)) \mapsto (x, z)$ continuous
   - Inverse $(x, y) \mapsto (y, x)$ continuous
   - $r, s : (x, y) \mapsto x, y$ (resp.) **local homeomorphisms**

... then take the associated (reduced) groupoid C^*-algebra, $C_r^*(R)$.

Preimages under an IFS

For $n \geq 1$, $x \in K$ let

$$\mathcal{F}^{-n}(x) = \{z \in K \mid \exists\ f_\xi = f_{\xi_1} \circ \cdots \circ f_{\xi_n} \text{ s.t. } f_\xi(z) = x\}.$$

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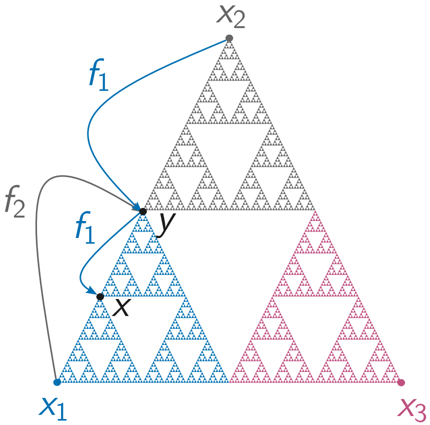

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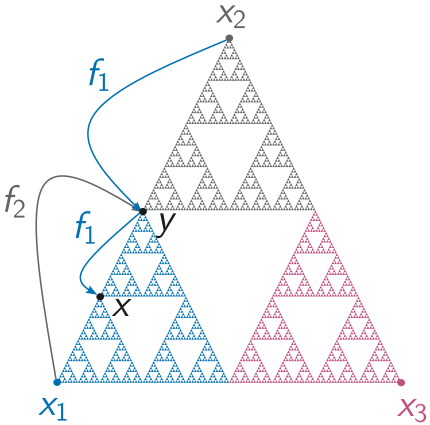

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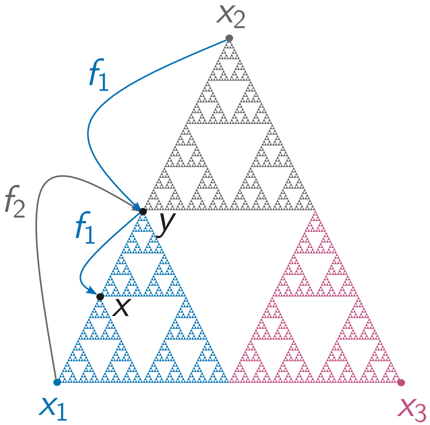

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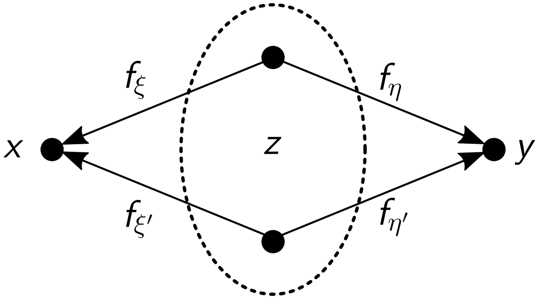
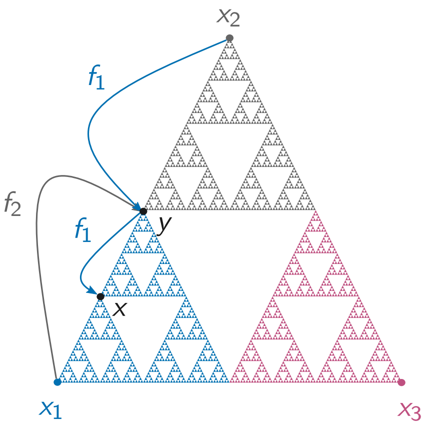
Ex. 2) $\mathcal{F}^{-2}(y) = \{x_1, x_2\}$.

Ex. 3) For all $n \geq 1$, $i \in \{1, 2, 3\}$, $\mathcal{F}^{-n}(x_i) = \{x_i\}$.

The Equivalence Relation

For each $n \geq 1$, define $R_n = \{(x, y) \in K \times K \mid \mathcal{F}^{-n}(x) = \mathcal{F}^{-n}(y)\}$.





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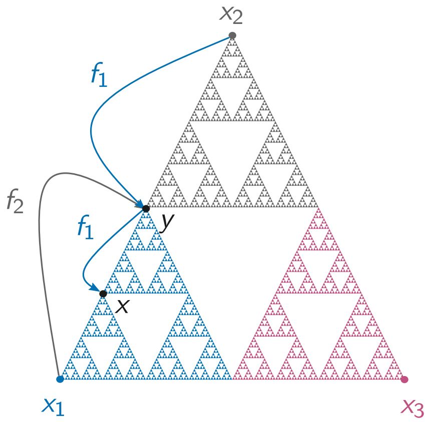

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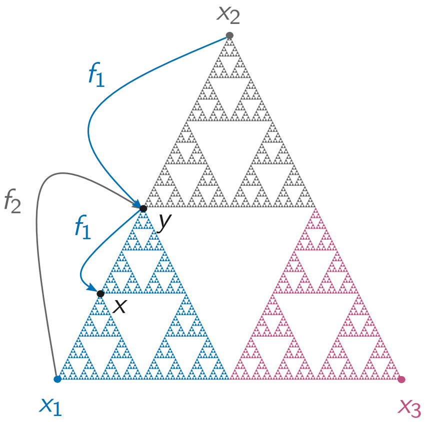

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- $x \sim_{R_2} y$.
- For all $n \geq 1$, $i \in \{1, 2, 3\}$, $[x_i]_{R_n} = \{x_i\}$.

The Equivalence Relation

$$R_1 \subseteq R_2 \subseteq R_3 \subseteq \cdots \subseteq \bigcup_{n \geq 1} R_n := R$$

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How to construct an étale topology for R?

Single-Matrix Affine IFS

Definition (SMAIFS)

Let $\mathcal{F} = \{f_1, \ldots, f_m\}$ be an IFS on $\mathbb{R}^d$. If there exists an $A \in \mathcal{M}_d(\mathbb{R})$ and $b_1, \ldots, b_m \in \mathbb{R}^d$ such that for each $i \in \{1, \ldots, m\}$

$$f_i(x) = Ax + b_i, \;\; x \in \mathbb{R}^d$$

then $\mathcal{F}$ is called a single-matrix affine IFS.

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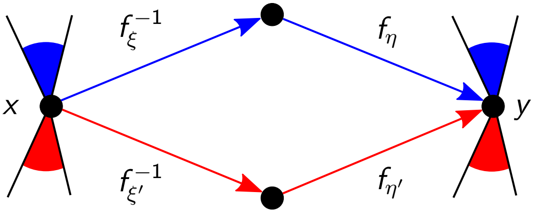
If $\mathcal{F}$ is an invertible SMAIFS (i.e. A is invertible), then

$$f_\eta \circ f_\xi^{-1} = (f_{\eta_1} \circ f_{\eta_2} \circ \cdots \circ f_{\eta_n}) \circ (f_{\xi_1} \circ f_{\xi_2} \circ \cdots \circ f_{\xi_n})^{-1}$$

is a **translation** for every $n \geq 1$.

A Local Action for R

If $\mathcal{F}$ is an invertible SMAIFS, then...



translation from x to y, $(x, y) \in R_n$, restricted to a small enough open set $U \subseteq K$, is a **partial homeomorphism** mapping every point $x' \in U$ to a point $y' \in [x']_{R_n}$.

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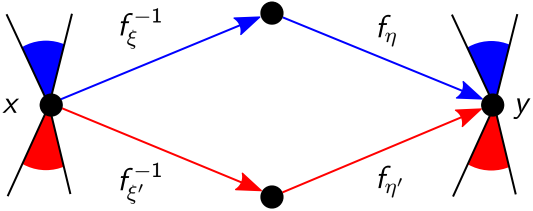

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The collection of these partial homeomorphisms over $n \geq 1$ forms a **basis** for an étale topology on R.

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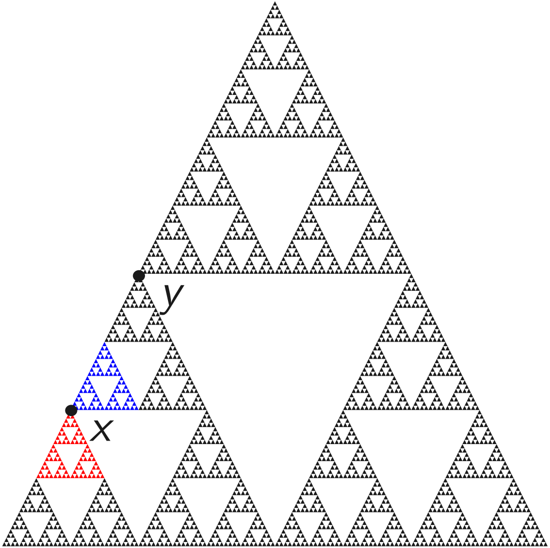

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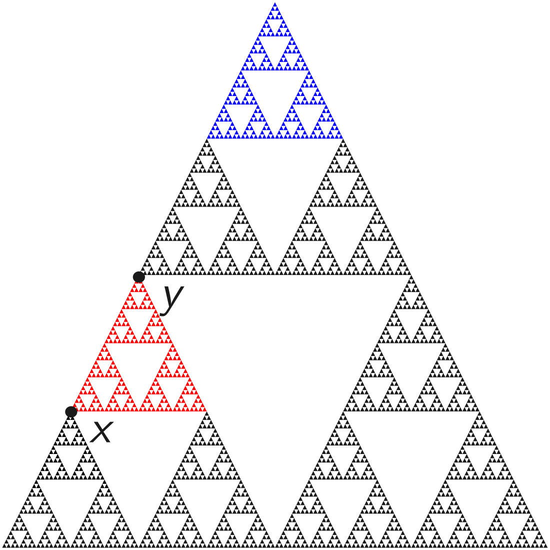

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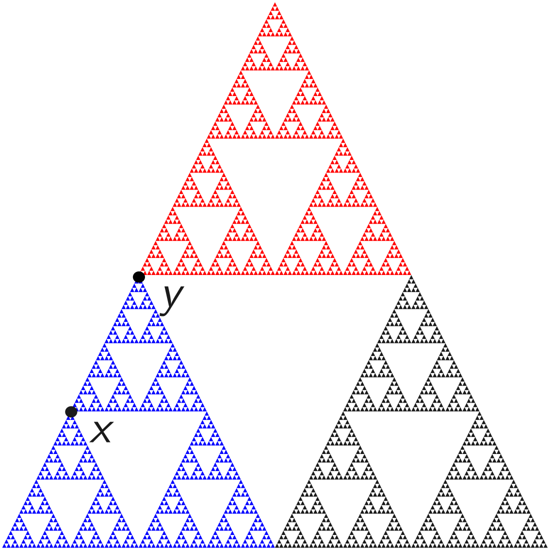

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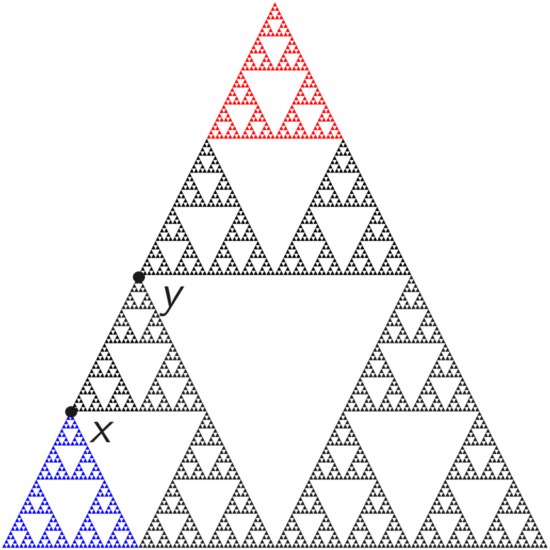

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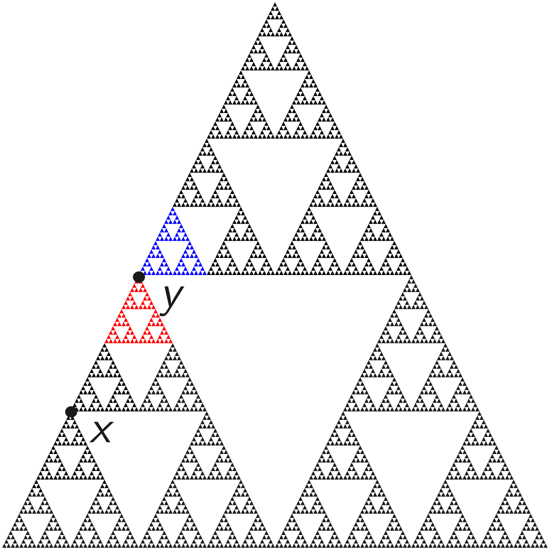


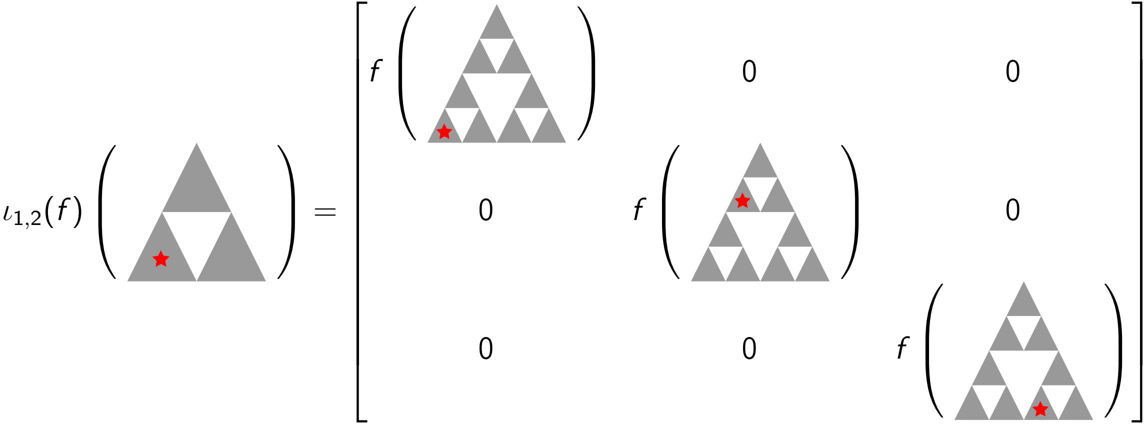
Inductive Limit Structure

$$R_1 \subseteq R_2 \subseteq \cdots \subseteq R_n \subseteq \dots \qquad R = \bigcup_{n \geq 1} R_n$$

$$C_r^*(R_1) \subseteq C_r^*(R_2) \subseteq \cdots \subseteq C_r^*(R_n) \subseteq \dots \qquad C_r^*(R) = \varinjlim C_r^*(R_n)$$

Inductive Limit Structure

For "typical points", $C_r^*(R_1) \xrightarrow{\iota_{1,2}} C_r^*(R_2)$ looks like...



A short-exact sequence arises from "non-typical points" for each $n \geq 1$:

$$0 \to C_0 \Bigg($$

$$\Bigg) \otimes \mathcal{M}_{3^n}(\mathbb{C}) \to C_r^* \Bigg(R_n \Big|$$

$$\Bigg) \to \bigoplus_{1}^{3} \mathcal{M}_{k_n}(\mathbb{C}) \to 0$$

where $k_n = 1 + 3 + 3^2 + \cdots + 3^{n-1}$.

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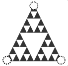
$\mathbb{Z} \xrightarrow{3} \mathbb{Z} \xrightarrow{3} \mathbb{Z} \xrightarrow{3} \mathbb{Z} \xrightarrow{3} \ldots$

$K_0\Big(C_r^*\Big(R\Big|$  $\Big)\Big) = \mathbb{Z}\left[\frac{1}{3}\right]$

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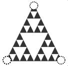
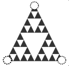
$K_0\Big(C_r^*\Big(R\Big|$  $\Big)\Big) = \mathbb{Z}\left[\frac{1}{3}\right]$

$$\mathbb{Z}^\infty \to \mathbb{Z}^\infty \to \mathbb{Z}^\infty \to \mathbb{Z}^\infty \to \dots$$

$K_1\Big(C_r^*\Big(R\Big|$  $\Big)\Big) = 0$