Leavitt path algebras as Cohn localisations

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Outline

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What is necessary for IBN violation?

Let R be a unital ring. Then $R \cong R^n$ as left modules iff there are elements $x_j, x_j^* \in R$ for $j \leq n$ such that

$$\sum_{j \leq n} x_j x_j^* = 1, \quad x_i^* x_j = \delta_{ij} \quad \forall i, j \leq n.$$

Proof.

Left module homomorphisms are of the shape

$$x\colon R \to R^n, \quad r \mapsto (rx_j)_{j \leq n};$$
$$x^*\colon R^n \to R, \quad (r_i)_{i \leq n} \mapsto \sum_{i \leq n} r_i x_i^*;$$

and inverse to each other iff the conditions above hold. □

Leavitt algebras

Let K be a field. Imposing these relations on a free associative K-algebra with generators X_j, X_j^* for $j \leq n$ yields

$$A := K\langle X_j, X_j^* \mid \sum_{j \leq n} X_j X_j^* = 1;\ X_i^* X_j = \delta_{ij};\ i,j \leq n\rangle$$

as a universal algebra with $A \cong A^n$ as left modules by construction.

Since William G. Leavitt has analysed them in 1962, these algebras are also known as Leavitt algebras.

Path algebras

Definition

Throughout, let K be a field and let $E := (E^0, E^1, s\colon E^1 \to E^0, r\colon E^1 \to E^0)$ be shorthand for a countable directed graph.

Let KE denote its path algebra, i.e. the free associative K-algebra generated by $E^0 \cup E^1$ subject to relations:

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$$(V) \quad vw = \delta_{v,w} v \quad \forall v, w \in E^0,$$
$$(E) \quad s(e)e = er(e) = e \quad \forall e \in E^1.$$

The set E^n of reduced words with n letters in E^1 consists of so called paths of length n.

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Note that $\mathrm{Path}(E) := \bigcup_{n \in \mathbb{N}_0} E^n$ forms a basis of KE and that paths are concatenated from left to right.

Vertex notation

Definition

A vertex $v \in E^0$ with

- $0 < |s^{-1}(v)| < \infty$ is called regular (otherwise singular). The set of regular vertices is called $\mathrm{Reg}(E)$.
- $s^{-1}(v) = \emptyset$ is a sink and the set of sinks is called $\mathrm{Sink}(E)$.
- $|s^{-1}(v)| = \infty$ is an infinite emitter and the set of infinite emitters is called $\mathrm{Inf}(E)$.

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If $\mathrm{Inf}(E) = \emptyset$, the graph E is called row-finite.
If $\mathrm{Reg}(E) = E^0$, the graph E is called regular.

Next goal: Localise KE

Remember Leavitt algebras $L(1, n)$:

- We demanded invertibility of a natural module homomorphism.
- In fact, we dealt with joint multiplication $(\cdot)X \colon K\langle X_1, \dots, X_n\rangle \to \bigoplus_{j \leq n} K\langle X_1, \dots, X_n\rangle \cdot X_j$.
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Now, carry this idea over to path algebras KE:

- Start with regular graphs.
- Include sinks and infinite emitters.

Cohn localisation in general

Definition

Let R be a ring and $(u_i\colon {}_RP^{(i)} \to {}_RQ^{(i)})_{i\in I}$ be a set of left R-module homomorphisms between projective and finitely generated R-modules.
The Cohn localisation of R at the maps $(u_i)_{i\in I}$ is the universal ring $\mathrm{Cohn}(R)$ with a homomorphism $R \to \mathrm{Cohn}(R)$ such that the induced maps

$$\mathrm{Cohn}(R) \otimes_R u_i\colon \mathrm{Cohn}(R) \otimes_R P^{(i)} \to \mathrm{Cohn}(R) \otimes_R Q^{(i)}$$

are left $\mathrm{Cohn}(R)$-module isomorphisms.

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By taking multiplication maps this generalises the classical localisation idea of making certain ring elements invertible.

. . . and in our context

- The Cohn localisation of an algebra A at $(u_i)_{i\in I}$ is defined analogously.
- If A has local units, any $u_i\colon P^{(i)} \to Q^{(i)}$ is a matrix multiplication map $m_i \in q_i M_{n_i}(A) p_i$ for certain idempotents p_i, q_i.
- $\mathrm{Cohn}(A)$ is obtained by adjoining all entries of the formal inverse matrices $m_i^* \in p_i M_{n_i}(\mathrm{Cohn}(A)) q_i$ with
  - $m_i^* m_i = p_i$,
  - $m_i m_i^* = q_i$.

Localisation at joint path prolongation maps

Given a regular graph E we want to localise $A := KE$ at the right multiplication maps

$$\varphi^v = (\cdot e)_{e \in s^{-1}(v)} \colon Av \to \bigoplus_{e \in s^{-1}(v)} Ar(e)$$

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for every $v \in E^0$. We enhance A to $\mathrm{Cohn}(A)$ such that the extended maps

$$\Phi^v \colon \mathrm{Cohn}(A)v \to \bigoplus_{e \in s^{-1}(v)} \mathrm{Cohn}(A)r(e)$$

are left $\mathrm{Cohn}(A)$-module isomorphisms. Their inverses

$$\Psi^v = \sum_{e \in s^{-1}(v)} \Psi^v_e \colon \bigoplus_{e \in s^{-1}(v)} \mathrm{Cohn}(A)r(e) \to \mathrm{Cohn}(A)v$$

are sums of right multiplication maps by elements

$$e^* := \Psi^v_e(r(e)) = r(e) \cdot \Psi^v_e(r(e)) \in r(e)\mathrm{Cohn}(A)s(e).$$

Cuntz-Krieger relations

- In short, the Cohn localisation of a regular path algebra at $(\cdot e)_{e \in s^{-1}(v)}$ is obtained by adjoining elements $\{e^* \mid e \in E^1\}$ such that the inverse homomorphisms are given by $\Psi^v = \sum_{e \in s^{-1}(v)} () \cdot e^*$.

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- They can be interpreted as additional ghost edges of an extended grpah $\hat{E} := (E^0, E^1 \cup (E^1)^*)$ with $s(e^*) := r(e)$, $r(e^*) := s(e)$ for $e \in E^1$.

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- Bijectivity is ensured by the Cuntz-Krieger relations:

$$\Psi^v \circ (\cdot e)_{e \in s^{-1}(v)} \colon \sum_{e \in s^{-1}(v)} ee^* = v \quad \forall v \in E^0,$$

$$(\cdot e)_{e \in s^{-1}(v)} \circ \Psi^v \colon \; e^* f = r(f)\delta_{e,f} \quad \forall e, f \in s^{-1}(v).$$

Leavitt path algebras

The previous slides motivate the following definition:

Definition

Let E be a graph and K a field. Impose the following Cuntz-Krieger relations on $K\hat{E}$ to get the Leavitt path algebra $L_K(E)$:

$$\begin{aligned} (CK1) \quad & e^*f = r(f)\delta_{e,f} \quad \forall e,f \in E^1, \\ (CK2) \quad & \sum_{e \in s^{-1}(v)} ee^* = v \quad \forall v \in \mathrm{Reg}(E). \end{aligned}$$

More generally, if one demands $(CK2)$ only for a subset $X \subset \mathrm{Reg}(E)$, this yields the relative Leavitt path algebra $L_K^X(E)$.

Connection to graph algebras

- $L_K(E)$ carries a canonical $*$-structure with self-induced vertices.
- $L_K(E)$ is the algebraic analogue of the graph C^*-algebra $C^*(E)$, which is the completion of $L_{\mathbb{C}}(E)$ in an appropriate norm topology that respects $ee^* \leq s(e)$.

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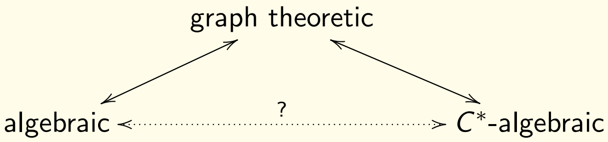
- Both algebra types share many properties that are, however, proven with different algebraic or topological arguments.
- Most relating results make use of the common graph theoretic structure.

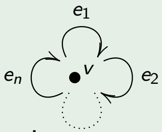


Leavitt algebras revisited

Example

These rose graphs for $n \geq 2$ produce the Leavitt algebras as their Leavitt path algebras:

$R_n =$



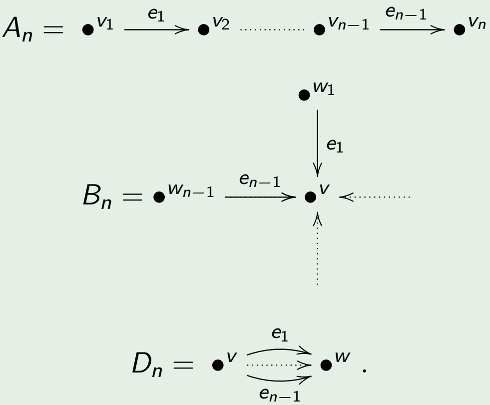
$L_K(R_1) \cong K[x, x^{-1}]$ gives the Laurent polynomials. The graph theoretic difference is that the cycle in R_1 has no exit, while R_n offers alternative cycles for $n \geq 2$.

For $K = \mathbb{C}$, the corresponding graph C^*-algebras are known as Cuntz algebras $\mathcal{O}_n$.

Matrix algebras

Example

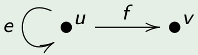
These graphs all produce matrix algebras $M_n(K)$ as their Leavitt path algebras:



Algebraic Toeplitz

Example

If we do not impose $(CK2)$ on the vertex in R_1, then we get $C_K^{\emptyset}(R_1)$, where s^* is only a left-inverse. Alternatively, also

$E_T =$ 

gives the algebraic Toeplitz algebra

$$L_K(E_T) \cong K\langle s, s^* \mid s^*s = 1\rangle$$

due to the properties of $S := e + f$ and $S^* := e^* + f^*$ in $L_K(E_T)$.
Note that the graph already suggests that this algebra has a matrix algebra and a Laurent part.

Q: How to deal with singular vertices?

$$\varphi^v = (\cdot e)_{e \in s^{-1}(v)} \colon KEv \to \bigoplus_{e \in s^{-1}(v)} KEr(e).$$

- For sinks $v \in \mathrm{Sink}(E)$, we would effectively set $v = 0$ and this is not desirable.
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- For infinite emitters $v \in \mathrm{Inf}(E)$, the target is not finitely generated and we cannot use it for a Cohn localisation map.
- Trick: Pass to a bigger algebra B that already contains the "source projections" $p_e = ee^*$:
  - Adjoin orthogonal idempotents $p_e p_f = \delta_{e,f} p_e \in vBv$ with $p_e e = e$ for $e, f \in s^{-1}(v)$.
  - This allows for a refinement to $(\cdot)e \colon Bp_e \to Br(e)$.

Q: How to deal with singular vertices?

Cohn localisation of B at the family of maps $(\cdot)e\colon Bp_e \to Br(e)$ for $e \in s^{-1}(v)$ means:

- We adjoin certain e^* such that the inverse is $(\cdot)e^*\colon \mathrm{Cohn}(B)r(e) \to \mathrm{Cohn}(B)p_e$.
- $ee^* = p_e$ and $e^*e = r(e)$.
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In the end, the generalised path algebra B is of similar complexity as kE itself and one can manage to show that it is quasi-free by lifting idempotents.
Furthermore, Cohn localisations preserve quasi-freeness and therefore we get that Leavitt path algebras are also quasi-free.

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$$a_1 d(a_2) + [A, \Omega^1(A)] \mapsto [a_1, a_2].$$

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- Canonical $\mathbb{Z}$-grading on A induced by path lengths. (Degree 1 for edges, degree 0 for vertices, degree -1 for ghost egdes).

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- Canonical $\mathbb{Z}$-grading on A induced by path lengths. (Degree 1 for edges, degree 0 for vertices, degree -1 for ghost egdes).
- Explicit direct sum decomposition of $\Omega^1(A)$.
- Construction of exact subcomplexes that simplify our problem.

Take-home message

- A Leavitt path algebra $L = L_K(E)$ models the algebraic core of a graph C*-algebra.
- Cuntz-Krieger relations arise as invertibility conditions for path prolongation maps.
- Justifies to treat L as Cohn localised path algebra $L = \mathrm{Cohn}(B)$.
- Key features for B pass on to L:
  - Convenient for quasi-free theory.
  - Simplifies the computation of $\Omega^1(L)$.
  - Beneficial for homological computations.

Bibliography

G. Abrams, P. Ara, M. S. Molina,
Leavitt Path Algebras,
Springer-Verlag London, 2017.

G. Abrams,
Leavitt path algebras: the first decade,
arXiv:1410.1835 [math.RA], 2014.

P. M. Cohn,
Introduction to Ring Theory,
Springer-Verlag London, Berlin, Heidelberg, 2000.

R. Meyer,
Noncommutative geometry,
NCG IV lecture notes, 2020.