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Equivariant $\mathcal{Z}$-stability for automorphisms

Lise Wouters[1]

9 August 2021

[1]Supported by a PhD fellowship of the Research Foundation Flanders (FWO)

Introduction

Central question

When is an action $\alpha : G \curvearrowright A$ of a countable group on a classifiable C^*-algebra **equivariantly $\mathcal{Z}$-stable**, i.e. when does it satisfy $\alpha \simeq_{\mathrm{cc}} \alpha \otimes \mathrm{id}_{\mathcal{Z}}$?

1. Motivation and statement of the problem
2. Nature of the problem
3. Known results
4. Proof techniques

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Importance: classification

Elliott program:

- Original goal to classify all simple, separable, nuclear C^*-algebras A
- Problem: $\mathrm{Ell}(\mathcal{Z}) = \mathrm{Ell}(\mathbb{C})$ and also $\mathrm{Ell}(A \otimes \mathcal{Z}) = \mathrm{Ell}(A)$

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Theorem (joint work of many researchers)

Let A and B be simple, separable, nuclear C^*-algebras that are $\mathcal{Z}$-stable and satisfy the UCT. Then

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$G \curvearrowright \bigotimes_{g \in G} \mathcal{Z}$ is equivariantly $\mathcal{Z}$-stable iff G is amenable

Traceless case

Theorem (Szabó, 2018)

Let A be a Kirchberg algebra and $\alpha : G \curvearrowright A$ an action by a countable amenable group. Then $\alpha \simeq_{\mathrm{cc}} \alpha \otimes \mathrm{id}_{\mathcal{O}_\infty}$.

$\rightarrow$ implies the result since $\mathcal{O}_\infty \cong \mathcal{O}_\infty \otimes \mathcal{Z}$

Traces on C^*-algebras

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Trace: $\tau : A \to \mathbb{C}$ pos. lin. functional of norm 1 with $\tau(ab) = \tau(ba)$

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- $||a||_{2,\tau} = \tau(a^*a)^{1/2}$ for $\tau \in T(A)$
- $||a||_{2,u} = \sup_{\tau \in T(A)} \tau(a^*a)^{1/2} = \sup_{\tau \in \partial_e T(A)} \tau(a^*a)^{1/2}$.

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$$\exists \text{ approximate } *\text{-homomorphisms } (M_2, \mathrm{id}_{\mathrm{M}_2}) \to (A, \alpha) \text{ that are}$$
$$\text{approximately unital, central and equivariant (w.r.t. } ||\cdot||_{2,u})$$

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$\exists$ approximate $*$-homomorphisms $(M_2, \mathrm{id}_{\mathrm{M}_2}) \to (A, \alpha)$ that are approximately unital, central and equivariant (w.r.t. $||\cdot||_{2,u}$)

For all $\varepsilon > 0$, finite $B \subset A$ and $F \subset G$ there exists a c.p.c. $\iota : M_2 \to A$ s.t.

- $||\iota(e_{i,j}e_{i',j'}) - \iota(e_{i,j})\iota(e_{i',j'})||_{2,u} < \varepsilon$
- $||\iota(1) - 1||_{2,u} < \varepsilon$
- $||[\iota(e_{i,j}), b]||_{2,u} < \varepsilon \; b \in B$
- $||\alpha_g(\iota(e_{i,j})) - \iota(e_{i,j})||_{2,u} < \varepsilon \; g \in F$.

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For each $\tau \in T(A)$ we can do this for $||\cdot||_{2,\tau}$ by known results about $\pi_\tau(A)''$.
$\to$ can we use this to obtain result for $||\cdot||_{2,u}$?

Known results (in presence of traces)

Theorem (Gardella–Hirshberg)

Let G be a countable amenable group, let A be a unital, separable, simple, nuclear, $\mathcal{Z}$-stable C^*-algebra and let $\alpha : G \curvearrowright A$ be an action. Suppose that $\partial_e T(A)$ is non-empty, compact and has finite covering dimension, and that the induced action of G on $\partial_e T(A)$ has finite orbits and Hausdorff orbit space. Then α is equivariantly $\mathcal{Z}$-stable.

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Equivariant $\mathcal{Z}$-stability for single automorphisms on simple C^*-algebras with tractable trace simplices:

Theorem (W)

Let A be a unital, separable, simple, nuclear and $\mathcal{Z}$-stable. Suppose that $\partial_e T(A)$ is non-empty, compact and has finite covering dimension. Then any automorphism $\alpha \in \mathrm{Aut}(A)$ is equivariantly $\mathcal{Z}$-stable.

Uniform tracial closure

$\bar{A}^u :=$ the C^*-algebra defined by adding limit points of $||\cdot||$-bounded, $||\cdot||_{2,u}$-Cauchy sequences.

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(Ozawa '13) For A unital, separable, simple, nuclear **with** $\partial_e T(A) \neq \emptyset$ **compact with finite covering dimension**, $\bar{A}^u$ is isomorphic to:

$C_\sigma(\partial_e T(A), \mathcal{R}) := \{f : \partial_e T(A) \rightarrow \mathcal{R} \mid ||\cdot||$-bounded and $||\cdot||_{2,\tau_{\mathcal{R}}}$-continuous.$\}$

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Conclusion

Problem reduces to a problem about actions $\alpha : G \curvearrowright C_\sigma(\partial_e T(A), \mathcal{R})$.

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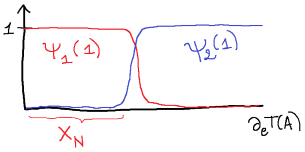
Choose ψ_1, ψ_2 such that

- $\psi_1(1)(\tau) = 1_{\mathcal{R}}$ for $\tau \in X_N$
- $\psi_2(1)(\tau) = 1_{\mathcal{R}}$ for $\tau \in (X_N)^{\mathrm{c}}$.



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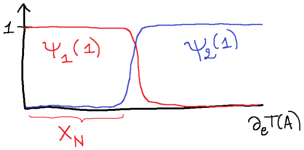

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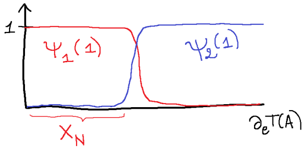

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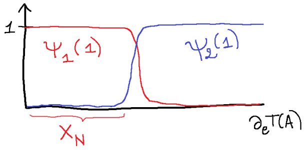

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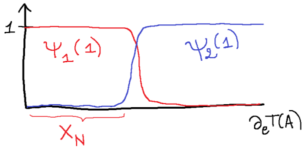

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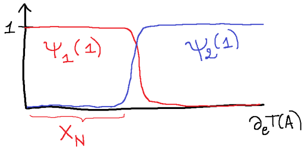

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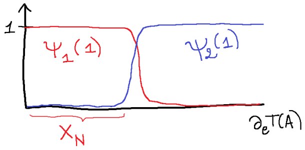

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Use towers to turn this into an approximately equivariant map ψ_2.

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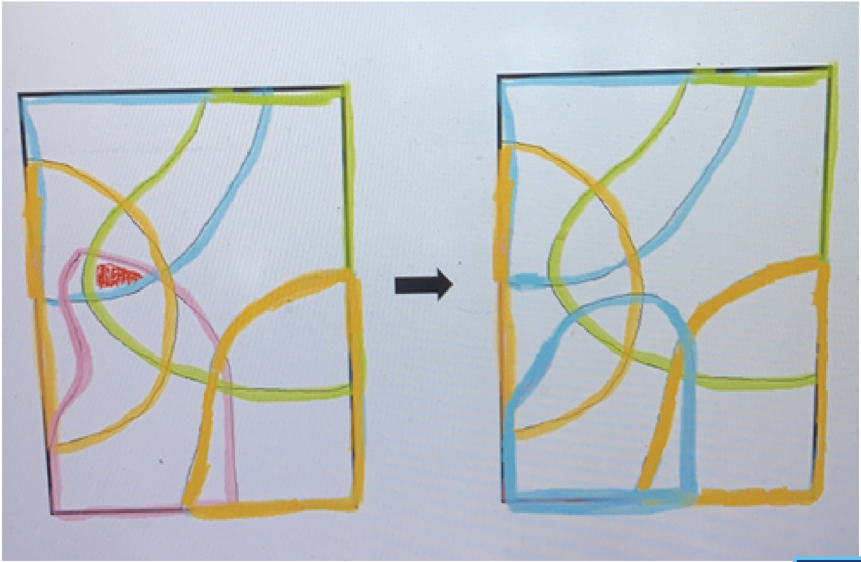
Future possibilities

- Obtain result for more general actions
- Different 'gluing' techniques (e.g. uniform property Γ and CPoU)

Thanks for your attention!

Finite covering dimension



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Theorem (W)

Let A be a unital, simple, separable, nuclear, $\mathcal{Z}$-stable C^*-algebra. Suppose that $\partial_e T(A)$ is non-empty, compact and finite-dimensional. Let $\mathcal{D}$ be a strongly self-absorbing C^*-algebra such that $A \cong A \otimes \mathcal{D}$. Then for any strongly outer action $\alpha : \mathbb{Z} \curvearrowright A$ and any $\delta : \mathbb{Z} \curvearrowright \mathcal{D}$ it holds that $\alpha \simeq_{\mathrm{cc}} \alpha \otimes \delta$.

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Idea: it suffices to show that for each unitary representation $\nu : \mathbb{Z} \to \mathcal{U}(M_n)$ there exist unital approximately central and equivariant $*$-homomorphism

$$(M_n, \mathrm{Ad}(\nu)) \to (A, \alpha)$$

This implies α has finite Rokhlin dimension with commuting towers and implies the result.