Lifts of completely positive equivariant maps

Marzieh Forough

Czech Academy of Sciences
based on joint work with Eusebio Gardella and Klaus Thomsen

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The completely positive lifting problem for C^*-algebras

Let A and B be C^*-algebras, and let I be a closed ideal of B with the quotient map $\pi\colon B \to B/I$. Suppose that $\varphi\colon A \to B/I$ be a completely positive contractive linear map.

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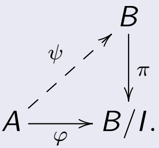

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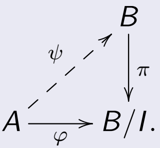

The importance of the existence of completely positive linear lifts can be traced back to Arveson's work on the extension theory.

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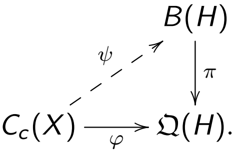

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Arveson used this result to give a simpler elegant proof of the Brown-Douglas-Fillmore theorem that $\mathrm{Ext}(C(X))$ is a group. In particular, Arveson showed that the availability of the lifting maps implies the existence of inverses in $\mathrm{Ext}(C(X))$.

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Motivated by the wide applications of the Choi-Effros lifting Theorem and the increased interest in the structure and the classification of C^*-dynamical systems, it is natural to investigate an equivariant version of the lifting result for completely positive maps.

1. G is a locally compact, second countable group;
2. (A, α) and (B, β) are G-algebras;
3. I is a G-invariant ideal in B with associated quotient map $\pi\colon B \to B/I$;
4. $\varphi\colon A \to B/I$ is equivariant completely positive contractive linear map.

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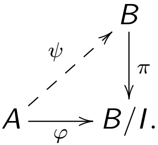

One may ask to determine conditions under which one can find an equivariant completely positive contractivelinear map $\psi\colon A \to B$ making the above diagram commutes.

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2. G is amenable.

Let $K \subseteq G$ and $F_A \subseteq A$ be finite sets and $\varepsilon > 0$.

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. Then for any $a \in F_A$ and $g \in K$, we get

$$\|\psi(\alpha_g(a)) - \beta_g(\psi(a))\| \leq \frac{2\|a\| |F_G \triangle g^{-1} F_G|}{|F_G|} \leq \varepsilon.$$

Therefore, for any compact set $K \subseteq G$, finite set $F \subseteq A$ and $\varepsilon > 0$, there exists a completely positive map linear $\psi\colon A \to B$ such that

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Instead of looking for an equivariant lift, we study the problem of finding "almost equivariant lift.

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Let $\varphi\colon A \to B/I$ be a completely positive contractive map which admits a completely positive linear lift $A \to B$.
Given $\varepsilon > 0$, a finite subset $F_A \subseteq A$, a compact subset $K \subseteq G$ there exists a completely positive contractive linear map $\psi\colon A \to B$ with $\pi \circ \psi = \varphi$, satisfying the following conditions for all $g \in K$ and $a \in F_A$:

$$\|(\psi \circ \alpha_g)(a) - (\beta_g \circ \psi)(a)\| \leq \|(\varphi \circ \alpha_g)(a) - (\overline{\beta}_g \circ \varphi)(a)\| + \epsilon$$

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Lemma 2

Let G be a locally compact, second countable group, let (M, γ) be a G-algebra and let I be a σ-unital G-invariant ideal in M. For every separable C^*-subalgebra $M_0 \subseteq M$, there exists a countable approximate unit $(x_n)_{n\in\mathbb{N}}$ in I satisfying the following conditions

(a) $x_n \leq x_{n+1}$ and $x_{n+1}x_n = x_n$ for all $n \in \mathbb{N}$;

(b) $\lim_{n\to\infty} \|x_n b - b x_n\| = 0$ for all $b \in M_0$;

(c) $\lim_{n\to\infty} \max_{g\in K} \|\gamma_g(x_n) - x_n\| = 0$ for all compact subsets $K \subseteq G$.

Notation

Let (I, γ) be a G-algebra, and let $M(I)$ be the multiplier algebra of I.

For each $g \in G$, the automorphism $\gamma_g \in \mathrm{Aut}(I)$ extends to an automorphism $\widetilde{\gamma}_g$ of $M(I)$, and the resulting assignment $\widetilde{\gamma}\colon G \to \mathrm{Aut}(M(I))$ is a group homomorphism.

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Since I is invariant by $\widetilde{\gamma}$, it follows that there is also a group homomorphism $\overline{\gamma}\colon G \to \mathrm{Aut}(Q(I))$ such that $\overline{\gamma}_g \circ q_I = q_I \circ \widetilde{\gamma}_g$ for all $g \in G$.

Proposition 3

Let G be a second countable, locally compact group, let (A, α) and (I, γ) be G-algebras, and assume that A is separable and I is σ-unital. Let $\varphi\colon A \to Q(I)$ and $\theta\colon A \to M(I)_{\widetilde{\gamma}}$ be completely positive contractions such that $q_I \circ \theta = \varphi$.

Given a finite subset $F \subseteq A$, a compact subset $K_G \subseteq G$, and $\varepsilon > 0$, there exists a completely positive contraction $\theta'\colon A \to M(I)_{\widetilde{\gamma}}$ with $q_I \circ \theta' = \varphi$, satisfying the following conditions for all $a \in F$:

$$\max_{g \in K_G} \left\| \widetilde{\gamma}_g(\theta'(a)) - \theta'(\alpha_g(a)) \right\| \leq \max_{g \in K_G} \left\| \overline{\gamma}_g(\varphi(a)) - \varphi(\alpha_g(a)) \right\| + \epsilon. \tag{1}$$

The idea of the proof of Proposition 2:

- Let $M_0 \subseteq M(I)_{\bar{\gamma}}$ be the separable G-algebra generated by $\theta(A)$. Let $(x_n)_{n\in\mathbb{N}}$ be an approximate identity for I as in the conclusion of Lemma 2 for M_0.
- Set $\Delta_0 = \sqrt{x_0}$ and $\Delta_n = \sqrt{x_n - x_{n-1}}$ for all $n \geq 1$.

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- Each θ_m is a lift for φ. The choice of (x_n) and the properties of Δ_n enable us to choose m such that θ_m satisfies in (1) for the given $F_A \subseteq A$, $K \subseteq G$ and $\varepsilon > 0$

The we use the Busby invariant for equivariant extensions of G-algebras to translate between the statement of Theorem 1 and that of Proposition 2.

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we show that there is a continuous family $\Theta = (\Theta_t)_{t\in[1,\infty)} : A \to B$ of lifts of φ, which are asymptotically linear, asymptotically completely positive, and asymptotically equivariant.

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(a) *unital*, if $\Theta_t(1) = 1$ for all t;

(b) *self-adjoint*, if Θ_t is self-adjoint for all $t \in [1,\infty)$;

(c) *asymptotically linear*, if

$$\lim_{t\to\infty} \|\Theta_t(\lambda_1 a_1 + \lambda_2 a_2) - \lambda_1\Theta_t(a_1) - \lambda_2\Theta_t(a_2)\| = 0$$

for all $\lambda_1, \lambda_2 \in \mathbb{C}$ and all $a_1, a_2 \in A$;

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$$\Theta \otimes \mathrm{id}_{M_n} = (\Theta_t \otimes \mathrm{id}_{M_n})_{t\in[1,\infty)} : M_n(A) \to M_n(B)$$

is asymptotically linear and asymptotically positive for all n.

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- ...

Theorem 4 (F-Gardella-Thomsen)

Let G be a second countable locally compact group, (A, α) and (B, β) be are G-algebras and I be a G-invariant ideal of B. Suppose that A is separable and $\psi\colon A \to B/I$ is a linear completely positive contractive map. There is an asymptotically equivariant lift $\Theta = (\Theta_t)_{t\in[1,\infty)}\colon A \to B$ of ψ,

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$$\lim_{t\to\infty} \|\Theta_t(a)\Theta_t(a)')\| = \|\psi(a)\psi(a')\|.$$

Thank you for listening!