$A\mathbb{T}$-algebras associated to zero-dimensional dynamical systems

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This was already done in the minimal case (Putnam '90) and the essentially minimal case with no periodic points (Herman-Putnam-Skau '92).

Zero-Dimensional Systems

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Examples of such X:

- Finite spaces with the discrete topology.
- Compactifications of the integers.
- The Cantor set.

Minimality

A dynamical system (X, h) is *minimal* if all orbits are dense; i.e. the only nonempty closed invariant subset of X is X.

Examples:

- $X = \mathbb{Z}/n\mathbb{Z}$ and h is the shift.
- X is the Cantor set and h is the 2-odometer.

Essential Minimality

A dynamical system (X, h) is *essentially minimal* if there is a unique smallest nonempty closed invariant subset of X. This smallest set is called the *minimal set*.

Examples:

- Any minimal system. The minimal set is X.
- X is the one-point compactification of the integers and h is the shift. The minimal set is the fixed point at infinity.

Crossed Products

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Given (X, h), we associate to it a *crossed product* C^*-algebra, which we will denote by $C^*(\mathbb{Z}, X, h)$, that "encodes the dynamical information" of the system. The most convenient way to think of it is the C^*-algebra generated by $C(X)$ and a unitary u such that

$$(u^* f u)(x) = f(h^{-1}(x)).$$

$A\mathbb{T}$-Algebras and Real Rank Zero

For our purposes, we will think of *$A\mathbb{T}$-algebras* as C^*-algebras that can be approximated arbitrarily well by circle algebras, which are direct sums of matrices and matrices over $C(S^1)$.

A$\mathbb{T}$-Algebras and Real Rank Zero

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Without giving a formal definition, we can consider *real rank zero* to be a condition on a C^*-algebra that tells us that there are "many projections".

The First Theorem

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Theorem (H. '21)

Let (X, h) be a zero-dimensional system that for any partition $\mathcal{P}$ of X, (X, h) admits a system of finite first return time maps subordinate to $\mathcal{P}$. Then $C^*(\mathbb{Z}, X, h)$ is an A$\mathbb{T}$-algebra.

Finite First Return Time Maps

Given (X, h) and a subset $Y \subset X$, we may define the "first return time map" $\lambda_Y : X \to \mathbb{Z}_{>0} \cup \{\infty\}$ by

$$\lambda_Y(x) = \inf\{n \in \mathbb{Z}_{>0} \mid h^n(x) \in Y\}$$

if the infimum exists, and $\lambda_Y(x) = \infty$ otherwise. This map tells us "how long it takes a point in Y to come back to Y".

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Some things to note:

- If X is minimal and Y is compact and open, then λ_Y is a finite subset of $\mathbb{Z}_{>0}$.
- If X is essentially minimal and Y is compact and open and intersects the minimal set of X, then λ_Y is a finite subset of $\mathbb{Z}_{>0}$.

An Example

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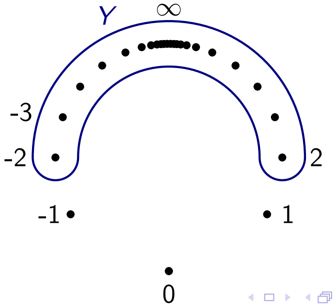
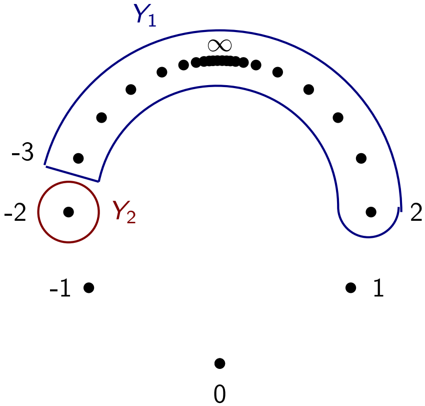
Consider the example of the one-point compactification of the integers with the shift homeomorphism.



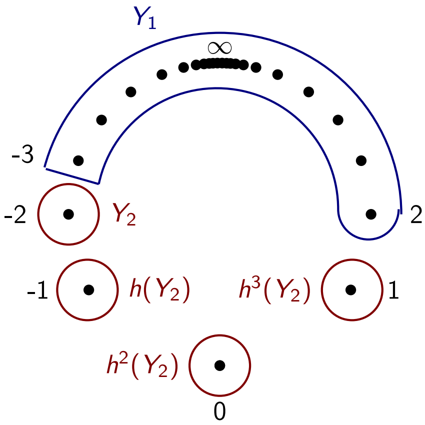
An Example

We split Y into pieces of constant first return time; the first return time of Y_1 is 1 and the first return time of Y_2 is 4.



An Example

We then use iterates to build a partition of X.

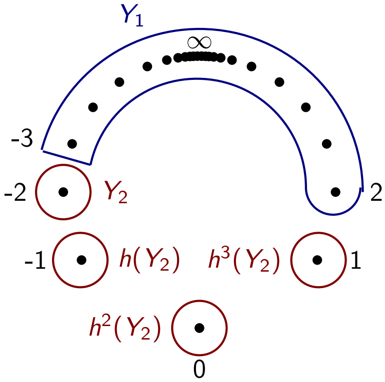


An Example

Using this partition, one can construct a matrix algebra that contains (as the diagonal matrices) an approximation of $C(X)$. The matrix units have the form

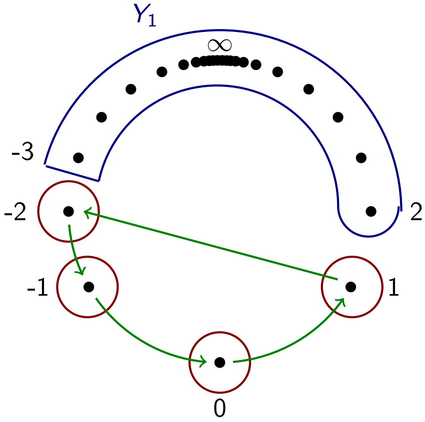
$$e_{i,j}^{(k)} = u^{i-j}\chi_{h^j(Y_k)}$$

and in our case is isomorphic to $\mathbb{C} \oplus M_4$.



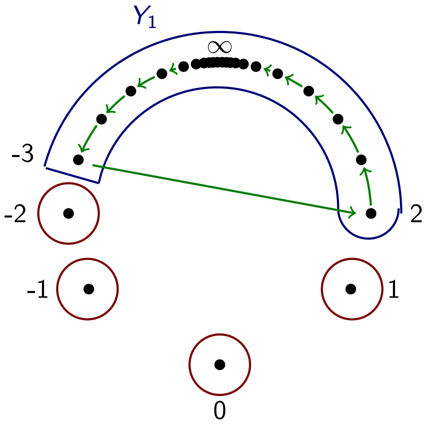
An Example

This matrix algebra does not approximate u very well. The best thing it has is the unitary $1 \oplus \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ that does the following:



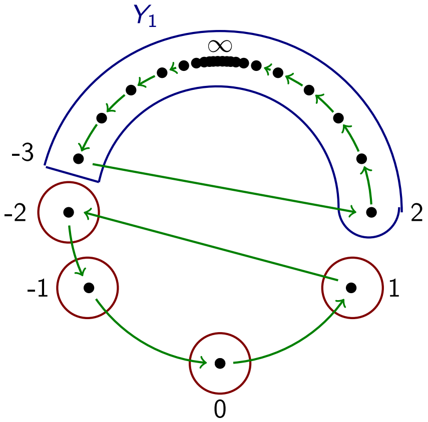
An Example

We then find a unitary in the crossed product that commutes with our matrix algebra that does the following:



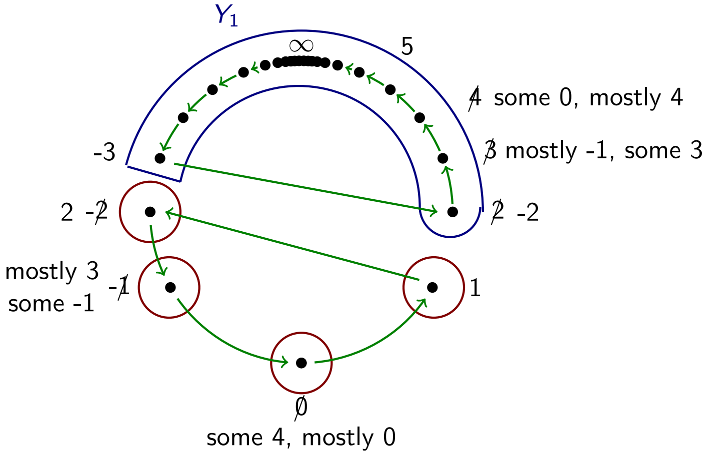
An Example

We then get a subalgebra $A \cong C(S^1) \oplus M_4$ that has a unitary $z \oplus \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ that does the following:



An Example

This still isn't quite right, so we use a unitary v to move some stuff. When done carefully, v^*Av will be a circle algebra that contains an approximation of u.



The Theorem Revisited

If one allows for finitely many such Y, this process still works and gives us an approximating circle algebra. Hence the previously referenced theorem:

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The Second Theorem

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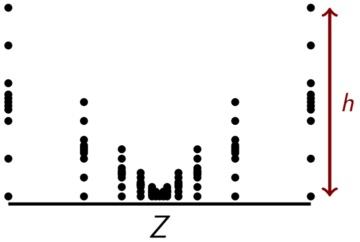
Let (X, h) be a zero-dimensional system. Then (X, h) is fiberwise essentially minimal if and only if for any partition $\mathcal{P}$ of X admits a system of finite first return time maps subordinate to $\mathcal{P}$.

An Example Generator

Theorem (H. '21)

Let (X, h) be a zero-dimensional system. Then (X, h) is fiberwise essentially minimal if and only if for any partition $\mathcal{P}$ of X admits a system of finite first return time maps subordinate to $\mathcal{P}$.

Without going into too much detail, *fiberwise essentially minimal* zero-dimensional look like this:



The Third Theorem

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Let (X, h) be a fiberwise essentially minimal zero-dimensional system with no periodic points. Then $C^*(\mathbb{Z}, X, h)$ has real rank zero.

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Altogether, we have shown that if (X, h) is fiberwise essentially minimal with no periodic points, then $C^*(\mathbb{Z}, X, h)$ is A$\mathbb{T}$ and has real rank zero, making it classifiable. Note that the first theorem actually didn't require the no periodic points assumption to yield A$\mathbb{T}$.