Quantum Graphs

(joint work with Michael Brannan and Samuel Harris)

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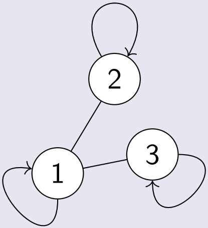
Classical Graphs

G = (Vertex set, Edge set, Adjacency matrix)

Example of a reflexive graph

- $V = \{1, 2, 3\}$
- $E = \{(1,1), (2,2), (3,3), (1,2), (1,3)\}$
- $A_G = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} * & * & * \\ * & * & 0 \\ * & 0 & * \end{bmatrix}$



$$S_G := \left\{ \begin{bmatrix} * & * & * \\ * & * & 0 \\ * & 0 & * \end{bmatrix} \text{ where } * \in \mathbb{C} \right\} \subseteq M_3(\mathbb{C})$$

S_G is an operator system !

Non-commutative graph associated with a classical graph

Suppose G is a classical graph with vertex set $V = \{1, 2 \ldots n\}$.

Graph Operator System

The non-commutative graph associated with the classical graph $G = (V, E)$ is the operator system S_G defined as

$$S_G = \text{ span}\{e_{ij} \ : \ (i,j) \in E \text{ or } i = j, \ \ \forall i, j \in V\} \subseteq \mathbb{M}_n \ ,$$

where e_{ij} are matrix units in $\mathbb{M}_n$.

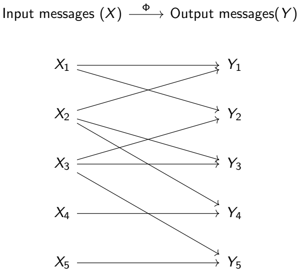
More generally,

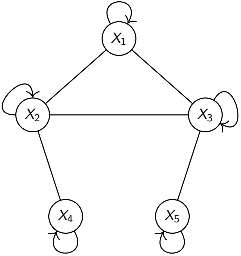
Matrix Quantum Graphs

An operator system in $\mathbb{M}_n$ is called a Matrix quantum graph.

Motivation from Information theory

- Used in zero-error communication.
- Generalize the confusability graph of classical channels.





A quantum channel is a completely-positive & trace-preserving linear map.

Non-commutative confusability graphs [DSW, 2010]

Given a quantum channel $\Phi : \mathbb{M}_m \to \mathbb{M}_n$ with $\Phi(x) = \sum_{i=1}^{r} K_i x K_i^*$, the confusability graph of Φ is the operator system:

$$S_\Phi = \text{ span}\{K_i^* K_j : 1 \leq i, j \leq r\} \subseteq \mathbb{M}_m.$$

Different approaches to quantum graphs

Classical graph $G = (V, E, A_G)$

- Quantize confusability graph of classical channels [DSW, 2010]
  - Matrix quantum graphs and Operator systems
  - Projection P_S onto the operator system S
- Quantize edge set $E \subseteq V \times V$ [Weaver, 2010, 2015]
  - Quantum relations
  - Projection P_E from χ_E
- Quantize adjacency matrix [MRV, 2018]
  - Categorical theory of quantum sets and quantum functions
  - Projection P_G using A_G

Unification

Under appropriate identifications, range of these projections is the same operator system!

Quantum graphs as quantum relations

Quantum set: von-Neumann algebra $\mathcal{M} \subseteq B(H)$.

Quantum relation [Weaver, 2010]

A quantum relation on $\mathcal{M}$ is a weak*-closed subspace $S \subseteq B(H)$ that is a bi-module over its commutant $\mathcal{M}'$.

Classical graph: $E \subseteq V \times V$ is a reflexive, symmetric relation on V.

Quantum Graph [Weaver, 2015]

A reflexive & symmetric quantum relation on $\mathcal{M}$ is a quantum graph.

$$_{\mathcal{M}'}S_{\mathcal{M}'} \underset{\text{operator system}}{\subseteq} B(H)$$

The quantum adjacency matrix formalism of quantum graphs

Quantum set: finite dimensional C*-algebra $\mathcal{M}$ with a fixed tracial state.

Quantum graph with a quantum adjacency matrix [MRV, 2018]

A quantum graph is a pair $(\mathcal{M}, A_G)$ containing

- Quantum set $\mathcal{M}$
- Quantum adjacency matrix $A_G : \mathcal{M} \xrightarrow{\text{linear}} \mathcal{M}$ with
  - Idempotency: $m(A_G \otimes A_G)m^* = A_G$
  - Reflexivity: $m(A_G \otimes I)m^* = I$
  - Symmetry: $(\eta^* m \otimes I)(I \otimes A_G \otimes I)(I \otimes m^* \eta) = A_G$

where m is the multiplication map and η is the swap map.

Comparing different notions of quantum graphs

Quantum set M: finite dimensional C*-algebra with fixed tracial state ψ.

CLASSICAL GRAPH	MATRIX Q.GRAPH	AS QUANTUM RELATIONS	AS PROJECTIONS	WITH ADJACENCY MATRIX
$G = (V, E, A_G)$ $A_G \in \mathbb{M}_n\{0, 1\}$	$S \subseteq \mathbb{M}_n$ is an operator system.	$(M, {}_{M'}S_{M'})$ weak*-closed operator sys in $B(H)$, bimodule over M'.	(M, p) $p \in M \otimes M^{op}$	(M, A_G) $A_G : M \to M$
Idempotency: $A_G \odot A_G = A_G$	$A_G \odot (\mathbb{M}_n) = S$	$M'SM' \subseteq S$	$p = p^* = p^2$	$m(A_G \otimes A_G)m^* = A_G$
Reflexivity: 1s on the diagonal	$1 \in S$	$M' \subseteq S$	$m(p) = 1_M$	$m(A_G \otimes I)m^* = I$
Undirected: $A_G = A_G^T$	$S = S^*$	$S = S^*$	$\sigma(p) = p$	$(\eta^* m \otimes I)(I \otimes A_G \otimes I)(I \otimes m^* \eta) = A_G$

Graph Homomorphisms

For classical graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$,

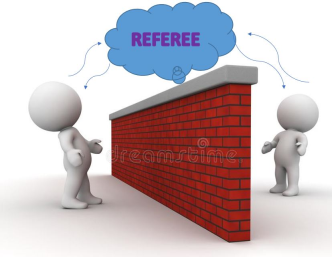
Classical Homomorphism

An adjacency preserving map $f : V_G \to V_H$,

$$(a, b) \in E_G \implies (f(a), f(b)) \in E_H.$$

Quantum Homomorphism (MR 2012)

Defined using a nonlocal game in which Alice and Bob try to convince a verifier with certainty that graph G admits a homomorphism to H.



Non-Local Graph Coloring Game

The quantum-to-classical graph homomorphism game

- Let G be a classical graph (simple, undirected) on c vertices.
- Let $(S, \mathcal{M}, M_n)$ be a quantum graph.
- $K_1 \oplus K_2 \oplus \ldots K_r = \mathbb{C}^n$ such that $\mathcal{M}$ acts irreducibly on each K_j.
- $\{v_1, v_2, \ldots v_n\} \underset{basis}{\subseteq} \mathbb{C}^n$ that can be partitioned into bases for $\{K_i\}_i^r$.

Definition (Brannan-G-Harris, 2020)

The **quantum-to-classical graph homomorphism game** *for* $((S, \mathcal{M}, M_n), G)$, *with respect to the basis* $\{v_1, ..., v_n\}$ *and a quantum edge basis* $\mathcal{F}$ *for* S *is:*

- *Inputs:* $\sum_{p,q} y_{\alpha,pq} \ \ v_p \otimes v_q$, *where* $Y_\alpha := \sum_{p,q} y_{\alpha,pq} \ \ v_p v_q^* \in \mathcal{F}$.
- *Outputs: vertices* $a, b \in \{1, 2, 3 \ldots c\} = V(G)$.
- *Winning Criteria:*
  - *Same vertex rule: If* $Y_\alpha \in \mathcal{M}'$, *then respond with same vertex* $(a = b)$.
  - *Adjacency rule: If* $Y_\alpha \perp \mathcal{M}'$, *then respond with adjacent vertices* $(a, b) \in E(G)$.

Connection to Stahlke's graph homomorphisms

Let $(S, \mathcal{M}, M_n)$ be a quantum graph & G be a classical graph on c-vertices.

Theorem [Brannan-G-Harris, 2020]

There is a winning quantum strategy for the graph homomorphism game for $((S, \mathcal{M}, M_n), G)$ if and only if there is a $d \in \mathbb{N}$ and a CPTP map $\Phi : \mathcal{M} \otimes M_d \longrightarrow D_c$ of the form $\Phi(.) = \sum_{i=1}^{m} F_i(.)F_i^*$ such that

$$F_i((S \cap (\mathcal{M}')^{\perp}) \otimes I_d)F_j^* \subseteq S_G \cap (D_c)^{\perp} \text{ for all } i, j.$$

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