Q-system completion for C*-algebras

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slides: https://people.math.osu.edu/hernandezpalomares.1/

Goals and overview

- ♣ Study *quantum symmetries* of C*-algebras:
  Unitary Tensor Categories (UTC) act on operator algebras via unitary tensor functors: [HHP20, Izu98, Jon20]

$$H : \mathsf{C} \overset{\otimes}{\hookrightarrow} \mathrm{Bim}(A).$$

- ♡ Perform *subfactor reconstruction* for C*-algs:
  all irreducible finite index extensions of II_1-factors are *crossed products* $N \subset N \rtimes_H Q$, where $Q \in \mathsf{C}$ is a *Q-system*. [JP19]
- ♠ C*-algs are *good receptacles* for UTC-actions:
  i.e. $\mathrm{C^*Alg}$ is *Q-system complete*. [CHPJP21]
- ♢ Induce new UTC-actions on W*/C*-algs from old. [GY20]

UTCs in nature

UTCs arise in various different contexts:

- Finite groups: $\mathrm{Hilb}(G, \omega)$, where $[\omega] \in H^3(G, U(1))$ determines associativity/coherence.
- Compact groups: Finite dimensional representations: $\mathrm{Rep}(G)$.
- Subfactors: the standard invariant of $N \subset M$ in terms of higher relative commutants.
- Discrete compact quantum groups: Tannaka-Krein duality: to $\mathbb{G}$ corresponds fiber functor $(\mathbf{F} : \mathrm{Rep}(\mathbb{G}) \to \mathrm{Hilb})$. [DCY13]

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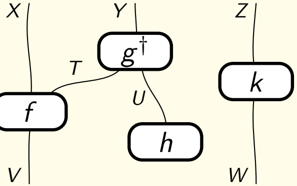
Graphical calculus for UTCs

- Diagrams read bottom to top
- Objects denoted by labelled strands
- 1-morphisms denoted by coupons:



$f : V \to X \otimes T$
$g : Y \to T \otimes U$
$h : 1_C \to U$
$k : W \to Z$

$- \circ -$ Composition by vertical stacking

$- \otimes -$ Tensoring by horizontal concatenation

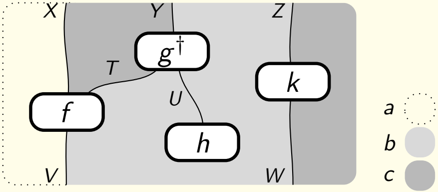
$\dagger$ Adjoint by vertical reflection

Graphical calculus for $\mathrm{C}^*/\mathrm{W}^*$-2-categories

▶ A tensor category is a 2-category with one object.

For 2-categories, we have a dimension shift:

- ○ Objects denoted by shadings,
- • 1-morphisms by strands,
- •• 2-morphisms by coupons.



$T : c \to b$
$U : b \to b$
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$W : b \to c$
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Example

There is a 2-category $\mathrm{C}^*\mathsf{Alg}$ whose objects are unital C^*-algebras, 1-morphisms are right Hilbert C^*-correspondences, and 2-morphisms adjointable intertwiners.

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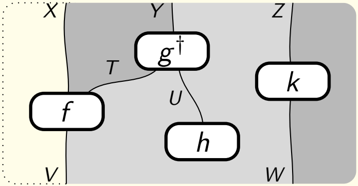

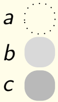

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C^*Alg : Right C*-correspondences in detail

C^*Alg is the C*-2-category consisting of:

- **0-mor:** Unital C*-algebras: A, B, C, ...
- **1-mor:** Right C*-Correspondences:
  ${}_AX_B \in C^*\mathrm{Alg}(A \to B), {}_BY_C \in C^*\mathrm{Alg}(B \to C), \ldots$
  A $\mathbb{C}$-vector space X with commuting left A- and right B-actions, and a right B-valued positive definite inner product:
  $$\langle \cdot | \cdot \rangle_B : \overline{X} \times X \to B.$$
  A left A-action on X by *adjointable operators*: A right B-linear map $T : X_B \to Z_B$ between right B-modules is *adjointable* if there is a right B-linear map $T^\dagger : Z_B \to X_B$ such that
  $$\langle \eta | T\xi \rangle_B = \langle T^\dagger \eta | \xi \rangle_B \qquad \forall \xi \in X, \forall \eta \in Z.$$
- **2-mor:** Adjointable intertwiners: $f \in C^*\mathrm{Alg}({}_AX_B \Rightarrow {}_AZ_B)$.

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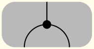
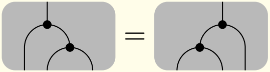
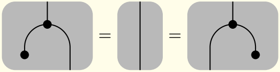
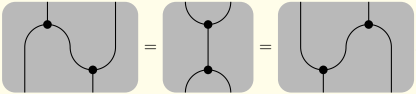
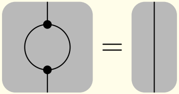
A **Q-system** in C is a 1-morphism $Q \in \mathrm{C}(b \to b)$ with multiplication $m =$  and unit $i =$  satisfying:

(Q1) Associativity:  ,

(Q2) Unitality:  ,

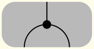
(Q3) Frobenius:  ,

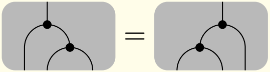
(Q4) Separable:  .[BKLR15]

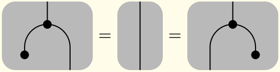
Remark

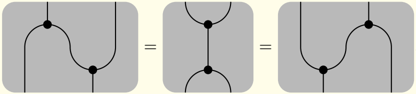
- Q-systems in UTCs give alternative axiomatization of the standard invariant of finite-index subfactors. [Müg03]
- Q-systems are also higher idempotents. Q-system completion for C*/W*-2-cats comparable with 2-cats of condensation monads. [DR18]

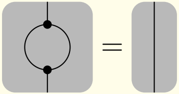
Q-systems in C*-2-categories

A **Q-system** in C is a 1-morphism $Q \in \mathrm{C}(b \to b)$ with multiplication $m =$  and unit $i =$  satisfying:

(Q1) Associativity: ,

(Q2) Unitality: ,

(Q3) Frobenius: ,

(Q4) Separable: .[BKLR15]

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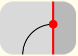
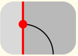
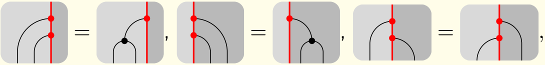
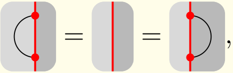
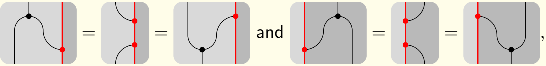
Bimodules over Q-systems

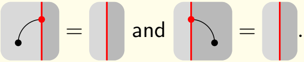
A bimodule $X \in \mathsf{C}(a \to b)$ over Q-systems $P \in \mathsf{C}(a \to a)$ and $Q \in \mathsf{C}(b \to b)$ consists of left and right actions

$\lambda =$  and $\rho =$ , satisfying

(B1) (associativity) 

(B2) (separable) 

(B3) (Frobenius) 

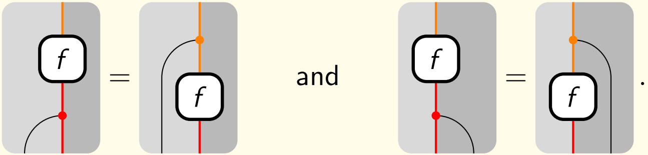
(B4) (unital) 

Bimodule intertwiners

Given Q-systems $P \in \mathsf{C}(a \to a)$ and $Q \in \mathsf{C}(b \to b)$, and $P - Q$ bimodules $X \in \mathsf{C}(a \to b)$ and $Y \in \mathsf{C}(a \to b)$, we define

$$\mathsf{QSys}(\mathsf{C})({}_P X_Q \Rightarrow {}_P Y_Q)$$

to consists of all those $f \in \mathsf{C}({}_a X_b \Rightarrow {}_a Y_b)$ such that



♣ This defines a C*-2-category $\mathsf{QSys}(\mathsf{C})$ with canonical embedding $\iota_{\mathsf{C}} : \mathsf{C} \to \mathsf{QSys}(\mathsf{C})$, mapping $\mathsf{C} \ni c \mapsto 1_c$, the trivial Q-system; i.e the monoidal unit ${}_Q Q_Q \in \mathsf{C}(Q \to Q)$.

▶▶▶ C is **Q-system complete**

$\overset{\text{Dfn}}{\Leftrightarrow}$ ι_{C} defines a †-2-equivalence

$\overset{\text{Thm}}{\Leftrightarrow}$ Q-systems unitarity split.

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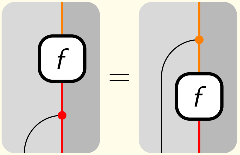
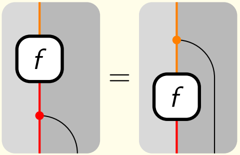


and



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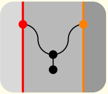
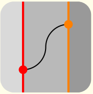
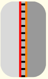
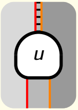
Composition of 1-morphisms

- Analogous to Connes fusion/Relative tensor product:

To compose the $P - Q$ bimodule ${}_A X_B$ and the $Q - R$ bimodule ${}_B Y_C$, we unitarily split the separability projector ([NY16])

$p_{X,Y} :=$  $:=$  $=$  $= u^\dagger_{X,Y} \circ u_{X,Y}$

for a coisometry $u_{X,Y}$, unique up to unique unitary.
Graphically:

 $= {}_A X \otimes_Q Y_C$ 

$= u_{X,Y}.$

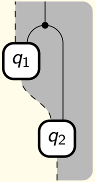
Realization of Q-systems

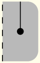
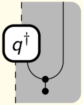
Theorem: [CHPJP21]

$\mathrm{C}^*\mathrm{Alg}$ is Q-system complete; i.e. $\mathrm{C}^*\mathrm{Alg} \cong \mathrm{QSys}(\mathrm{C}^*\mathrm{Alg})$.

Realization $|\cdot| : \mathrm{QSys}(\mathrm{C}^*\mathrm{Alg}) \to \mathrm{C}^*\mathrm{Alg}$ is inverse †-2-functor to $\iota_{\mathrm{C}^*\mathrm{Alg}} : \mathrm{C}^*\mathrm{Alg} \to \mathrm{QSys}(\mathrm{C}^*\mathrm{Alg})$, is defined as follows:

♠ A Q-system $Q \in \mathrm{C}^*\mathrm{Alg}(B \to B)$ maps to $|Q| := \mathrm{Hom}_{\mathbb{C}-B}(B \to Q)$:

$q_1 \cdot q_2 :=$ 

, $1_{|Q|} :=$ 
, $q^* :=$ 

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- $|Q|$ is C* via $|Q| \to \mathrm{End}_{-Q}(B \boxtimes_B Q)$, $\mathrm{End}_{-Q}(B \boxtimes_B Q) \to |Q|$

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mutually inverse unital $*$-isomorphisms.

Realization of Q-systems

Theorem: [CHPJP21]

$\mathrm{C^*Alg}$ is Q-system complete; i.e. $\mathrm{C^*Alg} \cong \mathrm{QSys}(\mathrm{C^*Alg})$.

Realization $|\cdot| : \mathrm{QSys}(\mathrm{C^*Alg}) \to \mathrm{C^*Alg}$ is inverse †-2-functor to $\iota_{\mathrm{C^*Alg}} : \mathrm{C^*Alg} \to \mathrm{QSys}(\mathrm{C^*Alg})$, is defined as follows:

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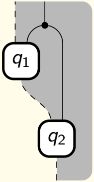
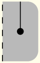
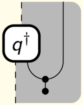
$q_1 \cdot q_2 :=$ 

, $1_{|Q|} :=$ 
, $q^* :=$ 

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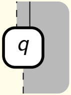
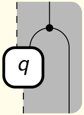
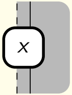
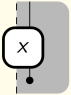


$\mapsto$ 



$\mapsto$ 

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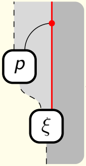
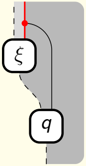
Realization of bimodules and intertwiners

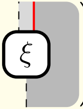
♠ P-Q bimod $X \in \mathrm{C^*Alg}(A \to B)$ gives $|X| := \mathrm{Hom}_{-B}(B \to X)$:

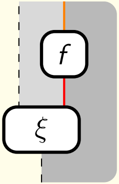
$p \rhd \xi :=$ 

, $\xi \lhd q :=$ 

$\forall f \in |P|$,
$\forall \eta \in |M|$, and
$\forall g \in |Q|$.

♠ $f \in \mathrm{C^*Alg}({}_AX_B \Rightarrow {}_AY_B)$ P-Q intertwiner maps to

$|f| : |X| \to |Y|$ given by $|f|\Bigg($ 

$\Bigg) :=$ 

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$|f|$ is $|P|$-$|Q|$ bimodular.

▶ Unitarily splitting separability projectors $p_{X,Y} = u^\dagger_{X,Y} \circ u_{X,Y}$ gives tensor structure for $|\cdot|$, and splitting of $1_{|Q|}$.

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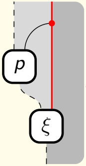
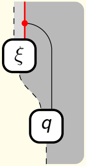
$p \triangleright \xi :=$ 

, $\xi \triangleleft q :=$ 

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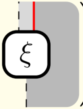
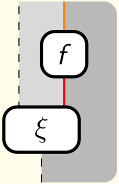
$|f| : |X| \to |Y|$ given by $|f|\left(\right.$ 

$\left.\right) :=$ 

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Conclusions and perspectives

♠ Realization splits the problem of classifying finite index extensions of a receptacle A in two parts:

(P1) *Analytical:* Constructing and classifying UTC-actions $H : \mathsf{C} \to \mathsf{Bim}(A)$.
Generalization of classification of groups actions on II_1 factors/C*-algebras, for which little is known for UTC.

(P2) *Algebraic:* Classifying Q-systems in a UTC.
Non-abelian cohomology problem.
Independent of A.

♣ $\mathrm{C}^*\mathsf{Alg}$ being Q-system complete allows for the straightforward adaptation of subfactor results to the C^*-setting; i.e. whenever C acts on A, we can automatically realize solutions to (P2) to obtain finite index extensions of A and bimodules between them.

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♢ Q-System completion induces new actions of UTCs Morita equivalent to C on finite extensions of A :

Example

For closed connected manifold X, $C(X)$ admits action from Hilb(G, ω). [Jon20]
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