Isometric Actions and Finite Approximations

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Finite approximation properties

- Finite (dimensional) approximation is ubiquitous in operator algebras (e.g. nuclear, MF, AF, quasidiagonal algebras)
- even common across other fields (e.g. polynomial estimates of functions, compactness)
- This talk will discuss finite approximation properties of group actions
- In particular, we will discuss a connection between group actions which fix a metric, and those which admit certain finite approximation properties.
- We'll also talk about some operator-algebraic implications of these properties

Residual finiteness (of groups)

- A group Γ is *residually finite* if for all $\gamma \in \Gamma$ there exists a normal subgroup $N < \Gamma$ of finite index such that $\gamma \notin N$
- so every element is 'seen' by some finite quotient
- examples: $\mathbb{Z}$, $GL_n(\mathbb{Z})$, free groups

Residual finiteness (of group actions)

- An action $\Gamma \curvearrowright X$ is *residually finite* if for all finite subsets $F \subset \Gamma$ and all $\epsilon > 0$ there is a finite set E, an action $\Gamma \curvearrowright E$, and a map $\zeta : E \to X$ such that $d(\zeta(\gamma \cdot e), \gamma \cdot \zeta(e)) < \epsilon$ for all $\gamma \in F$.
- Introduced in 2011 by Kerr and Nowak in [1] (although spiritually-similar definitions had been formulated in the past)
- This definition can be adapted to the non-metrizable setting (using neighborhoods of the diagonal), but we won't worry about that here

Examples

- $\mathbb{Z} \curvearrowright S^1$ by an irrational rotation – this can be shown directly using that the orbit of a point is dense
- If $\Gamma \curvearrowright E_n$ $(n = 0, 1, \ldots)$ is a sequence of actions on finite sets with equivariant connecting maps $E_{n+1} \to E_n$, we call the action $\Gamma \curvearrowright \varprojlim E_n$ on the inverse limit an Odometer. Odometers are residually finite (in fact, we will see something stronger is true).
- If F is a free group and $F \curvearrowright X$ fixes a measure of full support, then $\Gamma \curvearrowright X$ is residually finite ([1])

Basic Facts

- If X has no isolated points and $\Gamma \curvearrowright X$ is residually finite, we can assume ζ is an inclusion of spaces (still not equivariant), so $E \subset X$ (we'll assume this for the rest of the talk).
- Exercise: If $\Gamma \curvearrowright X$ free and residually finite, then Γ is residually finite (the freeness assumption is necessary, as any group admits an odometer action)
- Exercise: every RF action fixes a measure

Compact Groups

- If $\Gamma < G$ is finitely-generated and G is compact, then Γ is residually finite.
- This can be shown by using the representation theory of compact groups to map Γ into $n \times n$-matrices, and then showing that finitely-generated subgroups of such groups are residually finite
- Question: Is the action $\Gamma \curvearrowright G$ by translations residually finite? (the irrational rotation action on the circle discussed earlier is an example of one of these actions)

Translation actions

Theorem (P., 2021)

If $\Gamma < G$ is a finitely-generated, amenable subgroup of a compact group, then the action $\Gamma \curvearrowright G$ by translations is residually-finite.

Main ideas of proof

- The representation theory of compact groups lets us write G as an inverse limit of compact Lie groups, so we can assume G is a Lie group.
- Lie theory shows that an amenable, finitely-generated subgroup of a compact Lie group is virtually-abelian, so the closure of an orbit looks like some finite number of disjoint tori.
- A little Riemannian geometry and classifying isometries of flat tori allows us to construct approximations for such actions by virtually-abelian groups.
- I suspect the amenability assumption is unnecessary.

Isometric Actions

- If X is compact, $\mathrm{Isom}(X)$ is compact as well (by Arzelà-Ascoli), so if $\Gamma \curvearrowright X$ isometrically, then Γ is residually finite.
- Every isometric action fixes a measure of full support
- Question: Is every isometric action $\Gamma \curvearrowright X$ residually finite?

Isometric Actions on Cantor sets

Theorem (P., 2020)

If X is a Cantor set then $\Gamma \curvearrowright X$ fixes a metric iff it is conjugate (isomorphic) to an odometer (i.e. an inverse limit of Γ-actions on finite sets). If the original action is minimal, each level of the odometer is a transitive action.

Crossed-product C^*-algebras

Theorem (Kerr and Nowak, 2011)

Suppose $\Gamma \curvearrowright X$ is residually-finite. Then $C(X) \rtimes_r \Gamma$ is MF if $C_r^*(\Gamma)$ is MF, and is quasidiagonal if $C_r^*(\Gamma)$ is quasidiagonal.

Corollary

Suppose $\Gamma < G$ is a finitely-generated, amenable subgroup of a compact group. Then $C(G) \rtimes \Gamma$ is quasidiagonal.

Note: This corollary was already known as an application of [2, Corollary B] (and some other results), but the proof using residual-finiteness is more direct.

Crossed-product C^*-algebras (continued)

Corollary (P.)

Suppose X is a Cantor set and $\Gamma \curvearrowright X$ fixes a metric. Then $C(X) \rtimes_r \Gamma \cong \varinjlim \bigoplus_{\Gamma e \in \Gamma \backslash E_n} M_{|\Gamma e|}(C^*_r(\Gamma_{e,n}))$ (the groups $\Gamma_{e,n}$ are the stabilizer groups for $e \in E_n$ and the approximating action $\Gamma \curvearrowright E_n$).

David Kerr and P. Nowak.
Residually finite actions and crossed products.
Ergodic Theory and Dynamical Systems, 32:1585 – 1614, 2011.

Tikuisis, White, Winter.
Quasidiagonality of nuclear c^*-algebras.
Annals of Mathematics, 185:229–284, 2017.