Anomalous symmetries of simple operator algebras

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Outline

1. v.N algebraic background
2. C^*-results (joint work with Sam Evington)

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"classical" symmetries of $\mathcal{R}$

$$\mathcal{R} = \Big(\bigotimes_{i\in\mathbb{N}} (M_{n_i}(\mathbb{C}), \mathrm{tr})\Big)''$$

Classical symmetries.

- By self absorbtion (bernoulli shifts):

$$G \curvearrowright \otimes_G \mathcal{R} \cong \mathcal{R}.$$

- Through a representation $\rho : G \to M_n(\mathbb{C})$:

$$\mathrm{Ad}(\rho)^{\otimes\infty} : G \curvearrowright \mathcal{R}.$$

Classification of automorphisms

Definition

Let $\phi, \psi \in \mathrm{Aut}(\mathcal{R})$. We say they are conjugate if there exists $\theta \in \mathrm{Aut}(\mathcal{R})$ s.t. $\theta\phi\theta^{-1} = \psi$. We say they are outer conjugate if there exists $\theta \in \mathrm{Out}(\mathcal{R}) \cong \mathrm{Aut}(\mathcal{R})/\mathrm{Inn}(R)$ s.t. $\theta\overline{\phi}\theta^{-1} = \overline{\psi}$.

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Connes ('75, '77) classified automorphisms ϕ of $\mathcal{R}$ up to outer conjugacy. The invariant being:

- smallest $n \in \mathbb{N} \cup \{\infty\}$ s.t. $\overline{\phi}^n = \mathrm{id}$ in $\mathrm{Out}(\mathcal{R})$.
- If $\phi^n = \mathrm{Ad}(u)$, the n^{th} root of unity ω such that $\phi(u) = \omega u$.

The n^{th} root of unity

it follows that

$$\phi^{n+1} = \phi\phi^n = \phi \mathrm{Ad}(u) = \mathrm{Ad}(\phi(u))\phi$$

and

$$\phi^{n+1} = \phi^n\phi = \mathrm{Ad}(u)\phi$$

So $\phi(u) = \omega u$ for some ω in $Z(\mathcal{R}) = \mathbb{C}$.

$$u = \phi^n(u) = \dots\, \omega^n u$$

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Corollary

Connes construction also builds (n, ω) automorphisms on $M_{n^\infty} = \overline{\bigotimes M_n}^{\|.\|}$.

G-kernels

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Let $\overline{\theta}$ be a G-kernel on A then pick lifts θ_g and $u_{g,h}$ s.t.

$$\mathrm{Ad}(u_{g,h})\theta_g\theta_h = \theta_{gh} \tag{1}$$

Consider θ_{ghk}, use associativity of G to get

$$\theta_{g(hk)} = \mathrm{Ad}(u_{g,hk}\theta_g(u_{h,k}))\theta_g\theta_h\theta_k$$

$$\theta_{(gh)k} = \mathrm{Ad}(u_{gh,k}u_{g,h})\theta_g\theta_h\theta_k$$

So there exists $\omega(g,h,k) \in Z(U(A)) = \mathbb{C}$ such that $\omega(g,h,k) = u_{g,hk}\theta_g(u_{h,k})u^*_{g,h}u^*_{gh,k}$

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We call a lift $\{u : G \times G \to A,\ \theta : G \to \mathrm{Aut}(A)\}$ of a G-kernel with associated 3-cocycle $\omega \in Z^3(G, \mathbb{T})$ an ω-anomalous action.

G-kernels

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$$H^3(\mathbb{Z}/n\mathbb{Z}, \mathbb{T}) \cong \mathbb{Z}/n\mathbb{Z}.$$

Explicit representatives are given in terms of the n^{th} roots of unity.

Classification of G-kernels

As part of Jones classification of finite group actions on $\mathcal{R}$ ('80)

- Existence of G kernels for any countable discrete group G and $\omega \in H^3(G, \mathbb{T})$ on $\mathcal{R}$.
- Classification of injective G-kernels for G a finite group up to outer conjugacy by their class in H^3.

More generally one can look at actions of a unitary tensor categories on $\mathcal{R}$. An action of a unitary tensor category $\mathcal{C}$ on $\mathcal{R}$ is a fully faithful structure preserving functor

$$F : \mathcal{C} \to \mathrm{Bim}(\mathcal{R})$$

Existence+Uniqueness

It follows from Popa's results on classification of subfactors by their standard invariant ('94,'95) that one has:

1. Existence of actions for any unitary fusion category $\mathcal{C}$.
2. Uniqueness up to conjugacy of functors.

"Quantum" symmetries

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Examples

- $\mathcal{C} = \mathrm{Hilb}(G) \leftrightarrow$ cocycle G actions
- $\mathcal{C} = \mathrm{Hilb}(G, \omega) \leftrightarrow \omega$ anomalous actions
- $\mathrm{Rep}(G)$...

"Quantum" symmetries

Question I will address

How does the existence of G-kernels translate in the case of C^*-analogues of $\mathcal{R}$?

Outline

What C^*-algebras are we interested in?

"Classifiable"- for today $\mathcal{Z}$ and UHF.

Note

$A \cong A \otimes \mathcal{Z} \cong A \otimes \bigotimes_G \mathcal{Z}$

$$K_1^{\mathrm{alg}}(A) = \frac{U_\infty(A)}{[U_\infty(A), U_\infty(A)]}.$$

Note that

$$K_1^{\mathrm{alg}}(A) \twoheadrightarrow K_1(A)$$

Scalars can be non-zero in K_1^{alg}.

$\mathrm{K}_1^{\mathrm{alg}}(\mathbb{C})$

$K_1^{\mathrm{alg}}(\mathbb{C}) \cong \mathbb{T}$ and the isomorphism is induced by the determinant $\det : U_\infty(\mathbb{C}) \to \mathbb{T}$ whose kernel is $[U_\infty(\mathbb{C}), U_\infty(\mathbb{C})]$.

Algebraic K_1

This can be calculated for “classifiable” C*-algebras through the Skandalis de la Harpe determinant. For example

$$K_1^{\mathrm{alg}}(\mathcal{Z}) \cong \mathbb{R}/\mathbb{Z}$$
$$[e^{2\pi i h}] \underset{\Delta_{\mathcal{Z}}}{\rightarrow} \tau(h) + \mathbb{Z}$$

Algebraic K_1

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Observe

Suppose you have a unitary $u \in \mathcal{Z}$ such that $\theta(u) = \omega u$ then applying K_1^{alg} you get

$$K_1^{\text{alg}}(u) = K_1^{\text{alg}}(\theta(u)) = K_1^{\text{alg}}(\omega) + K_1^{\text{alg}}(u).$$

And so $\omega = 1$.

C^* obstruction results

Theorem 1 (Evington-G)

Let G be a group. Let ϕ be a G-kernel on $\mathcal{Z}$. Then $H^3(\phi)$ is trivial.

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$$K_1^{\mathrm{alg}}(M_{n^\infty}) \cong \frac{\mathbb{R}}{\mathbb{Z}[\frac{1}{n}]}.$$

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Theorem 2 (Evington-G)

Let ϕ be a G-kernel on the UHF algebra $\bigotimes_{k\in\mathbb{N}} M_{n_k}$. If $H^3(\phi)$ has finite order r then r divides $|G|$ and r^∞ divides $\prod_{k\in\mathbb{N}} n_k$.

C^* obstruction results

Theorem 1 (Evington-G)

Let ϕ be a G-kernel on the UHF algebra $\bigotimes_{k\in\mathbb{N}} M_{n_k}$. If $H^3(\phi)$ has finite order r then r divides $|G|$ and r^∞ divides $\prod_{k\in\mathbb{N}} n_k$.

- $K_1^{\mathrm{alg}}(\mathcal{R}) = 0$ **no obstruction**.

We can do things in more generality

- The results are not just ad hoc. We have a general obstruction arising from K_1^{alg}.
- We can also achieve existence results for these actions e.g. on UHF for arbitrary finite G and some C^*-algebras with non-trivial K_1 arising as crossed products by $\mathbb{Z}$ (e.g. Bunce Deddens, $\mathcal{A}_\theta$).

Questions?