Deformations of commuting squares and complex Hadamard matrices

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Hadamard Matrices

Definition

A unitary matrix H is Hadamard when every entry of H has the same Modulus. Equivalently, $H^*H = I$ and $|h_{i,j}| = \frac{1}{\sqrt{n}}$ for all $i, j \in \mathbb{Z}_n$.

- The Fourier Matrix F_n. Set ε to be the nth root of unity.

$$F_n = \frac{1}{\sqrt{n}} \left(\varepsilon^{ij}\right)_{i,j\in\mathbb{Z}_n}$$

- In particular, $n = 4$, $\varepsilon = i$

$$F_4 = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & 1 & i^2 \\ 1 & i^3 & i^2 & i \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$$

Why Study Hadamard Matrices?

H is a Hadamard Matrix. $\iff$

$$\mathfrak{C}(H) := \left(\begin{array}{ccc} \mathcal{D}_n & \subset & M_n(\mathbb{C}) \\ \cup & & \cup \\ \mathbb{C} & \subset & H\mathcal{D}_nH^* \end{array} , \quad \frac{1}{n}\mathrm{Tr} \right)$$

is a commuting square.

From any commuting square we can construct a subfactor by iterating Jones' Basic Construction:

$$\begin{array}{ccccccccc} \mathcal{D}_n & \subset & M_n(\mathbb{C}) & \subset^{e_3} & \mathcal{P}_2 & \subset^{e_4} & \dots & \subset & \overline{\bigcup \mathcal{P}_i} \\ \cup & & \cup & & \cup & & & & \cup \\ \mathbb{C} & \subset & H\mathcal{D}_nH^* & \subset^{e_3} & \mathcal{Q}_1 & \subset^{e_4} & \dots & \subset & \overline{\bigcup \mathcal{Q}_i} \end{array}$$

A Motivating Example

$$U(t) := \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & it & -1 & -it \\ 1 & -1 & 1 & -1 \\ 1 & -it & -1 & it \end{pmatrix}$$

- $U(t)$ is a Hadamard matrix for all $|t| = 1$.
- $U(1) = F_4$.

Question

What is the structure of the space of Hadamard matrices around the Fourier matrix?

The Defect of F_n

- Any analytic deformation of F_n yields a direction of convergence, which we can think of as a tangent to F_n.
- The dimension of T, the Tangent Space to F_n, is calculated in [NW20] as the defect of F_n or $d(F_n)$.

- Warning! It is not true in general that we can approach F_n from any direction in T.
- For $n = 6$ it remains an open problem whether or not you can approach F_6 from every direction in the tangent space. (On which dozens of papers have been published!)

Our Result

Theorem

Let n be a positive integer which has at least three distinct prime divisors. Then there exist h in the tangent space at F_n (of the algebraic manifold of Hadamard matrices), such that h is not a direction of convergence for any continuous family of complex Hadamard matrices containing F_n.

Proof Strategy:

- Take the "Second Derivative" of the relations. All directions in T automatically satisfy the second order relations. Main result of [NW14].
- Take the "Third Derivative" of the relations and find a direction of convergence in T which violates the 3rd order relations.

*This fact was conjectured to be true in [BeSB13].

Nuno Barros e Sá and Ingemar Bengtsson.
Families of complex Hadamard matrices.
Linear Algebra Appl., 438(7):2929–2957, 2013.

Remus Nicoara and Joseph White.
The defect of a group-type commuting square.
Rev. Roumaine Math. Pures Appl., 59(2):245–254, 2014.

Remus Nicoara and Joseph White.
Second order deformations of group commuting squares and Hadamard matrices.
Proc. Amer. Math. Soc., 148(9):3967–3974, 2020.