Orthogonality, Gateaux derivative and ideals in C^*-algebras

Sushil Singla

Shiv Nadar University (Delhi NCR), India.

Email id : ss774@snu.edu.in

1 Preliminaries

- In an inner product space V, v is orthogonal to a subspace W of V if and only if $\|v\| \leq \|v - w\|$ for all $w \in W$.
- **Birkhoff-James Orthogonality:** Let $(V, \|\cdot\|)$ be a normed space. Then a vector v is said to be Birkhoff-James orthogonal to a subspace W of V if $\|v\| \leq \|v - w\|$ for all $w \in W$.
- **Best approximation:** An element $w_0 \in W$ is said to be a best approximation to v in W if and only if $dist(v, W) = \|v - w_0\|$. Equivalently, $v - w_0$ is Birkhoff-James orthogonal to W.

So results about Birkhoff-James orthogonality give results for best approximations of a point to a subspace and vice versa.

Notations: Let $\mathcal{A}$ be a C^* algebra over field $\mathbb{F}$ (=$\mathbb{C}$ or $\mathbb{R}$). $\mathcal{S}_{\mathcal{A}}$ will stand for the set of all states on $\mathcal{A}$.

Gelfand-Naimark-Segal: Let $a \in \mathcal{A}$. Then there exists $\phi \in \mathcal{S}_{\mathcal{A}}$ such that $\phi(a^*a) = \|a\|^2$. This can be rephrased as : $\text{dist}(a, \{0\})^2 = \max\{\phi(a^*a) : \phi \in \mathcal{S}_{\mathcal{A}}\}$.

A positive functional ϕ gives a semi inner product on $\mathcal{A}$ defined as $\langle a_1|a_2\rangle_\phi = \phi(a_1^* a_2)$ for all $a_1, a_2 \in \mathcal{A}$. The above theorem can be rephrased as - There exists $\phi \in \mathcal{S}(\mathcal{A})$ such that $\|a\|_\phi = \text{dist}(a, \{0\})^2$.

We generalize this for any subspace $\mathcal{B}$, when a best approximation to a in $\mathcal{B}$ exists.

2 Orthogonality Characterizations

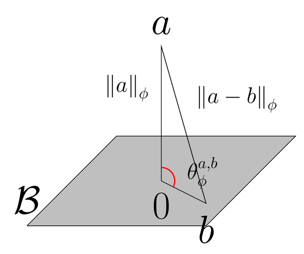
Theorem 2.1

Let $a \in \mathcal{A}$. Let $\mathcal{B}$ be a subspace of $\mathcal{A}$. Then a is Birkhoff-James orthogonal to $\mathcal{B}$ if and only if there exists $\phi \in \mathcal{S}_{\mathcal{A}}$ such that $\phi(a^*a) = \|a\|^2$ and $\phi(a^*b) = 0$ for all $b \in \mathcal{B}$.

The above theorem says that b_0 is a best approximation to a in $\mathcal{B}$ if and only if there exists $\phi \in \mathcal{S}_{\mathcal{A}}$ such that

$$\|a - b_0\|_\phi = \|a - b_0\| \text{ and } \langle a - b_0|b\rangle_\phi = 0 \text{ for all } b \in \mathcal{B}.$$

The above characterization of orthogonality has following geometric interpretation.



Special cases: Let $(\mathcal{C}(X), \|\cdot\|_\infty)$ be the space of $\mathbb{F}$-valued continuous functions on a compact Hausdorff space X. Let $\mathbb{M}_n(\mathbb{F})$ be the space of $n \times n$ matrices with entries in $\mathbb{F}$.

- **[Singer I., 1970]** Let $f \in \mathcal{C}(X)$ and W is a subspace of $\mathcal{C}(X)$. Let $g \in W$, then the following are equivalent:
  1. g is a best approximation to f in W.
  2. There exists a regular Borel probability measure μ on X such that
     (a) the support of μ is contained in the set $\{x \in X : |(f - g)(x)| = \|f - g\|_\infty\}$ and
     (b) $\int_X \overline{(f - g)} h \, d\mu = 0$ for all $h \in W$.

  Note that (a) is equivalent to $\int_X \overline{(f - g)}(f - g) \, d\mu = \|f - g\|_\infty^2$. And result follows using the Riesz representation theorem.
- **[Grover P., 2014]** Let $A \in \mathbb{M}_n(\mathbb{F})$ and $\mathcal{W}$ be a subspace of $\mathbb{M}_n(\mathbb{F})$. Then A is Birkhoff-James orthogonal to $\mathcal{W}$ if and only if there exists a positive semidefinite matrix $T \in \mathbb{M}_n(\mathbb{F})$ of trace one (also known as density matrix) such that $A^*AT = \|A\|^2 T$ and trace $(B^*AT) = 0$ for all $B \in \mathcal{W}$.

  We are using a simple fact here that for a positive semidefinite matrix T, $A^*AT = \|A\|^2 T$ is equivalent to trace $(A^*AT) = \|A\|^2 T$.
- **[Bhatia R.; Šemrl P., 1999]** A matrix A is orthogonal to B if and only if there exist unit vector x such that $\|Ax\| = \|A\|$ and $\langle Ax|Bx\rangle = 0$.

  This is an application of well celebrated Haudorff-Toeplitz theorem.
- **[Rieffel M. A., 2011]** Let $a \in \mathcal{A}$ be a Hermitian element and $\mathcal{B}$ be a C^*-subalgebra of a $\mathcal{A}$. If a is Birkhoff-James orthogonal to $\mathcal{B}$, then there exists $\phi \in \mathcal{S}_{\mathcal{A}}$ such that $\phi(a^2) = \|a\|^2$ and $\phi(ab + b^*a) = 0$ for all $b \in \mathcal{B}$.
- **[Arambašić L.; Rajić R., 2012]** Let $a \in \mathcal{A}$ and $\mathcal{B}$ be one dimensional subspace. Then a is Birkhoff-James orthogonal to $\mathcal{B}$ if and only if there exists $\phi \in \mathcal{S}_{\mathcal{A}}$ such that $\phi(a^*a) = \|a\|^2$ and $\phi(a^*b) = 0$ for all $b \in \mathcal{B}$.

Orthogonality in Hilbert C^*-module:
Let $\mathcal{E}$ be a Hilbert C^*-module over a complex C^*-algebra $\mathcal{A}$. It is known that $\mathcal{E}$ can be isometrically embedded into $\mathscr{B}(\mathcal{H}, \mathcal{K})$ for some Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$. Further, we can find a faithful representation $\pi : \mathcal{A} \to \mathscr{B}(\mathcal{H})$ and isometric embedding $L : \mathcal{E} \to \mathscr{B}(\mathcal{H}, \mathcal{K})$ satisfies $\langle L(e_1)h_1|L(e_2)h_2\rangle = \langle h_1|\pi(\langle e_1, e_2\rangle)h_2\rangle$ for all $e_1, e_2 \in \mathcal{E}$ and $h_1, h_2 \in \mathcal{H}$.

Theorem 2.2

Let $e \in \mathcal{E}$. Let $\mathcal{B}$ be a subspace of $\mathcal{E}$. Then e is Birkhoff-James orthogonal to $\mathcal{B}$ in the Banach space $\mathcal{E}$ if and only if there exists $\phi \in \mathcal{S}_{\mathcal{A}}$ such that $\phi(\langle e, e\rangle) = \|e\|^2$ and $\phi(\langle e, b\rangle) = 0$ for all $b \in \mathcal{B}$.

3 Proofs

Proof using representations of C^*-algebra

- A representation of $\mathcal{A}$ is a *-homomorphism $\pi : \mathcal{A} \to \mathscr{B}(\mathcal{H})$, for some Hilbert space $\mathcal{H}$.
- A representation $\pi : \mathcal{A} \to \mathscr{B}(\mathcal{H})$ is called cyclic if there exists a unit vector $\xi \in \mathcal{H}$ such that $\overline{\{\pi(a)\xi : a \in \mathcal{A}\}} = \mathcal{H}$.

Lemma: For a functional $\psi \in \mathcal{A}^*$, there exists a cyclic representation $(\mathcal{H}, \pi, \xi)$ and a vector $\eta \in \mathcal{H}$ such that $\psi(a) = \langle \eta|\pi(a)\xi\rangle$ for all $a \in \mathcal{A}$.

Using this, we give a proof of Theorem 2.1.

1. Let there exists $\phi \in \mathcal{S}_{\mathcal{A}}$ such that $\phi(a^*a) = \|a\|^2$ and $\phi(a^*b) = 0$ for all $b \in \mathcal{B}$. For all $b \in \mathcal{B}$,
$$\|a\|^2 = \phi(a^*a) \leq \phi(a^*a) + \phi(b^*b) = \phi((a - b)^*(a - b)) \leq \|a - b\|^2.$$
2. Let a be Birkhoff-James orthogonal to $\mathcal{B}$ i.e. $\text{dist}(a, \mathcal{B}) = \|a\|$.
3. By the Hahn-Banach theorem, there exists $\psi \in \mathcal{A}^*$ such that $\|\psi\| = 1$, $\psi(a) = \|a\|$ and $\psi(b) = 0$ for all $b \in \mathcal{B}$.
4. Hence there exists a cyclic representation $(\mathcal{H}, \pi, \xi)$ of $\mathcal{A}$ and a unit vector $\eta \in \mathcal{H}$ such that
$$\psi(c) = \langle \eta|\pi(c)\xi\rangle \text{ for all } c \in \mathcal{A}.$$
5. Now $\psi(a) = \langle \eta|\pi(a)\xi\rangle = \|a\|$. So by using the condition for equality in the Cauchy-Schwarz inequality, we obtain $\|a\|\eta = \pi(a)\xi$.
6. This gives $\psi(c) = \dfrac{1}{\|a\|}\langle \pi(a)\xi|\pi(c)\xi\rangle$ for all $c \in \mathcal{A}$.
7. Therefore, $\langle \pi(a)\xi|\pi(a)\xi\rangle = \|a\|^2$ and $\langle \pi(a)\xi|\pi(b)\xi\rangle = 0$ for all $b \in \mathcal{B}$.
8. Define $\phi \in \mathcal{A}^*$ as $\phi(c) = \langle \xi|\pi(c)\xi\rangle$.

Proof using convex analysis

We consider the function $f(\lambda) = \|a + \lambda b\|$ mapping $\mathbb{C}$ into $\mathbb{R}_+$. To say that a is orthogonal to b is to say that f attains its minimum at the point 0. This is clearly a calculus problem, except that the function $\|\cdot\|$ is not differentiable.

So we can't use first derivative test. But $\|\cdot\|$ is also convex, this gives motivation to define Gateaux derivative and we will see characterization of orthogonality in terms of Gateaux derivative. We have following theorem using convexity of the norm function.

Theorem 3.1

Let X be a Banach space, $x, y \in X$, and $\phi \in [0, 2\pi)$.

1. The function $\alpha : \mathbb{R} \to \mathbb{R}$ defined by, $\alpha(t) = \|x + ty\|$ is convex. Hence the limit $D_{0,x}(y) = \lim_{t\to 0^+} \dfrac{\|x + ty\| - \|x\|}{t}$ always exists.
2. We have $\|x + ty\| \geq \|x\|$ for all $t \in \mathbb{R}$ if and only if the inequality $D_{0,x}(y) \geq 0$ holds.
3. And x is orthogonal to y if and only if $\inf_\phi D_{\phi,x}(y) \geq 0$ where $D_{\phi,x}(y) = \lim_{t\to 0^+} \dfrac{\|x + te^{i\phi}y\| - \|x\|}{t}$ is called the ϕ-Gateaux derivative of the norm at the vector x, in the y and ϕ directions.

Using the C^*-identity, we can prove the following lemma.

Lemma 3.1

Let $a, b \in \mathcal{A}$. Then
$$\lim_{t\to 0^+} \frac{\|a + tb\| - \|a\|}{t} = \frac{1}{\|a\|} \lim_{t\to 0^+} \frac{\|a^*a + ta^*b\| - \|a^*a\|}{t}.$$

Proof of Theorem 2.1.

Using the above lemma, we get $D_{\phi,a}(b) = \dfrac{1}{\|a\|} D_{\phi,a^*a}(a^*b)$. This gives $\inf_\phi D_{\phi,a}(b) \geq 0$ if and only if $\inf_\phi D_{\phi,a^*a}(a^*b) \geq 0$ i.e. a is orthogonal to b if and only if a^*a is orthogonal to a^*b.

By the Hahn-Banach Theorem, there exists $\phi \in \mathcal{A}^*$ such that $\|\phi\| = 1$, $\phi(a^*a) = \|a\|^2$ and $\phi(a^*b) = 0$ for all $b \in \mathcal{B}$. And ϕ is the required state using fact it attains its norm at a non-zero positive element. □

4 Applications

For a normed space V and $v, u \in V$, we have
$$\lim_{t\to 0^+} \frac{\|v + tu\| - \|v\|}{t} = \max\{\text{Re } f(u) : f \in V^*, f(v) = \|v\|, \|f\| = 1\}.$$

Theorem 4.1

Let $a, b \in \mathcal{A}$. Then we have
$$\lim_{t\to 0^+} \frac{\|a + tb\| - \|a\|}{t} = \frac{1}{\|a\|} \max\{\text{Re}\phi(a^*b) : \phi \in \mathcal{S}_{\mathcal{A}}, \phi(a^*a) = \|a\|^2\}.$$

It is well known that if $\mathcal{I}$ is a two-sided closed ideal of $\mathcal{A}$, then any $\phi \in \mathcal{S}_{\mathcal{I}}$ has a unique extension $\tilde{\psi} \in \mathcal{S}_{\mathcal{A}}$. We will use the same notation ψ for the extension $\tilde{\psi}$ throughout the article. And $\mathcal{A}^* = \mathcal{I}^\# \oplus_1 \mathcal{I}^\perp$, where $\mathcal{I}^\# = \{f \in \mathcal{A}^* : \|f\| = \|f|_{\mathcal{I}}\|\}$. Using these facts, we get the following generalization of above theorem.

Theorem 4.2

Let $\mathcal{I}$ be a two-sided closed ideal of $\mathcal{A}$. Let $a \in \mathcal{A} \setminus \{0\}$ be such that $\text{dist}(a, \mathcal{I}) < \|a\|$. Then for any $b \in \mathcal{A} \setminus \{0\}$, we have
$$\lim_{t\to 0^+} \frac{\|a + tb\| - \|a\|}{t} = \frac{1}{\|a\|} \max\{\text{Re } \phi(a^*b) : \phi \in \mathcal{S}_{\mathcal{I}}, \phi(a^*a) = \|a\|^2\}.$$

Let $\mathscr{B}(\mathcal{H})$ and $\mathscr{K}(\mathcal{H})$ denotes the space of bounded and compact operators on a Hilbert space $\mathcal{H}$, respectively.

Corollary. For $A \in \mathscr{B}(\mathcal{H})$ be such that $\text{dist}(A, \mathscr{K}(\mathcal{H}) < \|A\|$. Then for any $B \in \mathscr{B}(\mathcal{H})$, we have
$$\lim_{t\to 0^+} \frac{\|A + tB\| - \|A\|}{t} = \frac{1}{\|A\|} \max_{\|u\|=1, A^*Au=\|A\|^2 u} Re\langle Au|Bu\rangle.$$

Definition 4.1

We say that a vector v of norm one is a smooth point of the unit ball of V if there exists a unique functional F_v, called the support functional, such that $\|F_v\| = 1$ and $F_v(v) = 1$.

It is a general fact that v is a smooth point of the unit ball of v if and only if $\lim_{t\to 0} \dfrac{\|v + tu\| - \|v\|}{t}$ exists and in this case, it is equal to Re $F_v(u)$. Also, if A is a smooth point, then it can not attain its norm at more than one point and if $\text{dist}(A, \mathscr{K}(\mathcal{H})) = \|A\|$, then A is not a smooth point of the unit ball of $\mathscr{B}(\mathcal{H})$.

Corollary 4.1

An operator A is a smooth point of the unit ball of $\mathscr{B}(\mathcal{H})$ if and only if A attains its norm at a unit vector h such that $\sup_{x\perp h, \|x\|=1} \|Ax\| < \|A\|$. In that case, $\lim_{t\to 0} \dfrac{\|A + tB\| - \|A\|}{t} = Re\langle Ah|Bh\rangle$.

Finally, we end with a non trivial result.

Theorem 4.3

Let $A \in \mathscr{B}(\mathcal{H}, \mathcal{K})$ be such that $\text{dist}(A, \mathscr{K}(\mathcal{H}, \mathcal{K})) < \|A\|$. Let $\mathcal{B} \subseteq \mathscr{B}(\mathcal{H}, \mathcal{K})$ be a subspace. Then A is orthogonal to $\mathcal{B}$ if and only if there exists at most countable set $\mathcal{J}$, a set of positive numbers $\{s_j : j \in \mathcal{J}\}$ and an orthonormal set $\{u_j \in \mathcal{H} : j \in \mathcal{J}\}$ such that

(i) $\sum_{j \in \mathcal{J}} s_j = 1$,

(ii) $A^*Au_j = \|A\|^2 u_j$ for each $j \in \mathcal{J}$,
and

(iii) $\sum_{j \in \mathcal{J}} s_j \langle Au_j | Bu_j \rangle = 0$ for all $B \in \mathcal{B}$.

References

1. Grover P. ; Singla S. : Best Approximations, distance formulas and orthogonality in C^*-algebras. *J. Ramanujan Math. Soc.* 36 (2021), 85–91.

2. Grover P. ; Singla S. : Birkhoff-James orthogonality and applications : A survey. *Operator Theory, Functional Analysis and Applications*, Birkhäuser, Springer, vol. 282, 2021. https://doi.org/10.1007/978-3-030-51945-2_15

3. Singla S. : Gateaux derivative of C^* norm. *Linear Algebra Appl.*. (to appear) https://doi.org/10.1016/j.laa.2021.07.019